# NEVANLINNA THEORY FOR THE DIFFERENTIAL - DIFFERENCE MONOMIALS AND DIFFERENCE POLYNOMIALS

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**Abstract**

The purpose of this paper is to extend the Nevanlinna Theory to the theory for the monomials containing $f(z)$, its shifts, derivatives and derivatives of its shifts. The relation between characteristic function and deficiencies of the function and its differential-difference monomials is studied and we also investigate the uniqueness of a function sharing small functions with its difference polynomials in f.

**AMS Subject Classification: 30D35, 30D30, 39A10.**

***Key words:*** *Meromorphic function; Differential- difference monomial; Difference polynomials; deficiency.*

**Introduction and Definitions**

Nevanlinna's theory of value distribution is concerned with the density of points where a meromorphic function takes a certain value in the complex plane and one of the advantages of this theory is that it applies not only to entire functions, but to meromorphic functions as well. One of the early results in this area is Picard's theorem which states that a non-constant entire function can omit at most one value. Nevanlinna offered a deep generalization of Picard's theorem in the form of his second fundamental theorem (Hayman, 1964) which implies the defect relation:

$$\sum_{a} (\delta(a,f)+\theta(a,f)) \leq 2$$

Where the sum is taken over all points in the extended complex plane, and the quantities $\delta(a,f)$ and $\theta(a,f)$ are called the **deficiency** and the **index of multiplicity** of the value **a**, respectively.

**DEFINITION 1**: The Nevanlinna deficiency is defined as

$$\delta(a,f) = \lim_{r\to\infty}.\inf \frac{m(r,a)}{T(r,f)},$$

Where 'a' is in the extended complex plane. The ramification index or index of multiplicity is

$$\theta(a,f)=\liminf_{r\to\infty}\frac{N(r,a)-\overline{N}(r,a)}{T(r,f)},$$

Where N(r, a) is the counting function of the a- points of f, counting multiplicities, and $\overline{N}(r,a)$ the counting function ignoring multiplicities.

**DEFINITION 2:** We say that the two meromorphic functions f and g **share** the complex value **a CM (IM),** if (f – a) and (g – a) have the same zeros **counting multiplicities (ignoring multiplicities).** In the sequel, a meromorphic function a(z) is said to be **a small function** of 'f' if and only if T(r, a(z)) = o(T(r, f)), where r approaches to infinity outside a possible exceptional set of finite logarithmic measure.

Another consequence of Nevanlinna's second fundamental theorem is the five value theorem, which says that if two non- constant meromorphic functions share five values ignoring multiplicity then these functions must be identical.

To study this paper, the reader should be familiar with the basic concepts of Nevanlinna theory [ 1 ] such as the Characteristic function $T(r,f)$, proximity function $m(r,f)$, counting function $N(r,f)$ and so on. In addition, $S(r,f)$ denotes the quantity that satisfies the condition that $S(r,f)=o(T(r,f))$, as $r\to\infty$ outside a possible exceptional set of finite logarithmic measure.

By a formal discretization of the derivative function f ′(z), we obtain

$$\frac{f(z+c)-f(z)}{c}=\frac{\Delta_c f}{c}, \text{ where } c \in \mathbb{C}.$$

**DEFINITION 3**: For a meromorphic function $f$, we define its **shift** by $f_c(z)=f(z+c)$, and its **difference operators** by

$$\Delta_c f=f(z+c)-f(z),$$

$$\Delta_c^n f=\Delta_c^{n-1}(\Delta_c f)=\sum_{k=0}^{n}\frac{(-1)^k.n!}{k!(n-k)!}f(z+(n-k)c) \quad \ldots(1)$$

for all $n\geq \square, n\geq 2$. In particular, $\Delta_c^n f(z)=\Delta^n f(z)$ for c = 1. From (1), we see that the difference operators of f are the linear combination of its shifts. $\Delta_n^c f$ is called a **difference polynomial in f**

**DEFINITION 4:** A general **Differential- Difference Monomial** is a product containing powers of f (z), its shifts, its derivatives and derivatives of its shifts i.e. an expression of the form

$$M[f] = \begin{array}{l} a(z)(f(z))^{p_{0,0}}(f'(z))^{p_{0,1}}\ldots(f^m(z))^{p_{0,m}} \\ (f(z+c_1))^{p_{1,0}}(f'(z+c_1))^{p_{1,1}}\ldots(f^m(z+c_1))^{p_{1,m}} \\ \cdots \\ (f(z+c_n))^{p_{n,0}}(f'(z+c_n))^{p_{n,1}}\ldots f^m(z+c_n))^{p_{n,m}} \end{array} \quad \ldots(2)$$

Then the degree of M[f] will be the sum of all the powers in the product on the right side of above expression. From (2), we see that any shift of 'f' is a particular case of M[f].

**Preliminaries**

1. Uniqueness theory of meromorphic functions is an important part of Nevanlinna theory. Recently, there has been an increasing interest in studying difference equations and shifts of meromorphic functions. Many authors investigated the uniqueness of meromorphic functions with shifts, monomials, difference polynomials etc. and deficiencies of difference analogue (Dhar, 2016; Yang and Laine, 2007; Yang and Laine, 2010; Zheng and Chen, 2011). Halburd and Korhonen (2006a, 2006b) established a version of Nevanlinna theory based on difference operators and shifts and they proved the difference analogue on the logarithimic derivative which is one of the main ingredients in studying difference equations and shifts of functions of finite order. Heittokangas (2009, 2011) proved the uniqueness results for shifts of a meromorphic functions as follows:

**THEOREM A:** Let $f$ be a meromorphic function of finite order, and let $c \in \mathbb{C}$, then

a. If $f(z)$ and $f(z+c)$ share three distinct periodic small functions with period c CM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.

b. If $f(z)$ and $f(z+c)$ share two distinct periodic small functions with period c CM, and if $\overline{\lim_{r\to\infty}} \frac{N(r,f)}{T(r,f)} < 1$, for all $z \in \mathbb{C}$.

c. If $f(z)$ and $f(z+c)$ share $\infty$ CM and a periodic small functions a (z) with period c

CM, and if there exists a periodic small function b (not equal to a(z)) such that $\overline{\lim_{r\to\infty}}\frac{N(r,b,f)}{T(r,f)}<1$, then $f(z)=f(z+c)$ for all $z\in\mathbb{C}$.

It is well known that the Valiron- Mokhon'ko Theorem is of essential importance in the theory of complex differential equations and functional equations. The difference analogue of this theorem with some additional assumptions has been proved by Zheng and Chen (2011). They proved that:

**THEOREM B:** Suppose that P[z, f] is a difference polynomial of the form

$$P[z,f]=\sum a_{\lambda}f^{i_0}(f(z+c_1))^{i_1}\dots(f(z+c_n))^{i_n},$$

where $f(z)$ is a transcendental meromorphic function of finite order satisfying

$$N(r,f)+N(r,0,f)=S(r,f), \text{ then}$$

$$T(r,P)=d.T(r,f)+S(r,f).$$

**Main Results**

In this paper, we consider a **monomial** containing shifts, derivatives and derivatives of the shifts of a non-constant meromorphic function of finite order and any shift or derivative of the function is a particular case of that monomial. We prove the following results:

**THEOREM 1:** Let $f$ be a non- constant transcendental meromorphic function of finite order with $\theta(\infty,f)=1$. If M [$f$] is a differential- difference monomial (as in (2)) of degree d, then

$$\overline{\lim_{r\to\infty}}\frac{T(r,M[f])}{T(r,f)}\le d, \text{ and } \overline{\lim_{r\to\infty}}\frac{T(r,\Delta_c^n f)}{T(r,f)}=n+1.$$

**THEOREM 2:** Let $f$ be a meromorphic function of finite order with $\theta(0,f)+\theta(\infty,f)=2$. Then $\delta(\alpha,M[f])=0$, where $\alpha$ is a non-zero, finite small function and $M[f]$ is a monomial of the form (2), i.e. M [$f$] - $\alpha$ has infinitely many zeros.

**THEOREM 3:** Let $f$ be a meromorphic function of **finite** order, and let $c\in\mathbb{C}$, then for non-constant $\frac{\Delta_n^c[f]}{f}$, $\overline{N}(r,f)=S(r,f)$, $N(r,0,f)<T(r,f)$, then if $f(z)$ and $\Delta_n^c[f]$

share one small non- zero finite function IM, then $f(z) \equiv \Delta_n^c f$.

In order to prove above theorems, we need the following lemmas:

**LEMMA 1[2]:**

Let $f$ be a meromorphic function of finite order, then for $c \in \mathbb{C}$,

i. $$m(r, \frac{f(z+c)}{f(z)}) = S(r,f), \quad m(r, \frac{f(z)}{f(z+c)}) = S(r,f)$$

ii. $$T(r, f(z+c)) = T(r,f)$$
$$N(r, f(z+c)) = N(r,f) + S(r,f)$$
$$N(r, \frac{1}{f(z+c)}) = N(r, \frac{1}{f}) + S(r,f)$$

**LEMMA 2[2]:**

Let $f$ (z) be a non – constant meromorphic solution of $(f(z))^n P(z,f) = Q(z,f)$, where $P(z,f)$ and $Q(z,f)$ are difference polynomials in $f$ (z) and degree of $Q(z,f)$ as a polynomial in $f$ (z) and its shifts is at most n, then

$$\mathrm{m}(\mathrm{r}, P(z,f)) = S(r,f),$$

For all r outside of a possibly exceptional set with finite logarithmic measure.

**Proof of Theorems**

**PROOF OF THEOREM 1:** Using Lemma1 and $\theta(\infty, f) = 1$, we get

$$\begin{aligned}
& T(r, \mathrm{M}[f]) \\
& = m(r, \mathrm{M}[f]) + N(r, \mathrm{M}[f]) \\
& = m(r, \mathrm{M}[f]) + \mathrm{S}(r,f) \\
& \leq m(r, \frac{\mathrm{M}[f]}{f^d}) + d.T(r,f) + \mathrm{S}(r,f) \\
& = d.T(r,f) + \mathrm{S}(r,f)
\end{aligned}$$

and thus

$$\overline{\lim_{r\to\infty}}\frac{T(r,\mathrm{M}[f])}{T(r,f)}\le d.$$

For second part, we have

$$\begin{aligned}&T(r,\Delta_c^n f)\\&=T(r,\sum_{k=0}^{n}\frac{(-1)^k.n!}{k!(n-k)!}f(z+(n-k)c)\\&=(n+1).T(r,f)+\mathrm{S}(r,f)\end{aligned}$$

and thus

$$\overline{\lim_{r\to\infty}}\frac{T(r,\Delta_c^n f)}{T(r,f)}=n+1.$$

This proves Theorem 1.

**PROOF OF THEOREM 2:** By using Nevanlinna's Second Main Theorem, Lemma 1, Theorem 1, and the condition $\theta(0,f)+\theta(\infty,f)=2$, we get

$$\begin{aligned}&T(r,M[f])\\&\le\overline{N}(r,M[f])+\overline{N}(r,\frac{1}{M[f]})+\overline{N}(r,\frac{1}{M[f]-\alpha})+S(r,M[f])\\&\le\overline{N}(r,\frac{1}{M[f]-\alpha})+S(r,M[f])\end{aligned}$$

and hence $\delta(\alpha,\mathrm{M}[f])=0$.

This proves Theorem 2.

**PROOF OF THEOREM 3:** Suppose on the contrary, the assertion that $\Delta_n^c[f]\neq f(z)$. Let $f(z)$ and $\Delta_n^c[f]$ share one small non- zero finite function as a(z).

Using Nevanlinna's Second Fundamental Theorem, Lemma 1 and given conditions, we get

$$T(r,f) \leq \overline{N}(r,0,f)+\overline{N}(r,f)+\overline{N}(r,a(z),f)+S(r,f)$$

$$\leq \overline{N}(r,\frac{\Delta_n^c[f]}{f},1)+S(r,f),$$

$$\leq T(r,\frac{\Delta_n^c f}{f})+S(r,f)$$

$$= N(r,\frac{\Delta_n^c f}{f})+S(r,f)$$

$$\leq N(r,\frac{1}{f})+S(r,f)$$

$$< T(r,f)+S(r,f),$$

which is a contradiction, and therefore, $\Delta_n^c[f] \equiv f(z)$.

**EXAMPLES:**

1. Let c be a non – zero complex number, and let $f(z)=e^{\sin\frac{\pi z}{c}}$. Clearly $f$ is of infinite order of growth, here $\overline{N}(r,f)=S(r,f)$, $N(r,0,f) < T(r,f)$, here $f$ (z) and $f$ (z + c) share 1, yet $f$ (z+c) ≠ $f$ (z). Hence the condition that the **finiteness of order** of function cannot be dropped in Theorem 3.

2. Let $f(z)=1+e^{2\pi iz}$, c=1 then $f$ and $\Delta^n f$ share 1 , but $\Delta^n f \neq f$. The reason being that the condition $N(r,0,f) < T(r,f)$ is not satisfied and therefore, cannot be dropped in Theorem 3.

3. Let the function $f(z)=e^z+1$, $M[f] = f(z).f(z+\pi i)$ then $M[f] - 1$ has no zeros. The reason is that $f(z)$ has infinitely many zeros and thus $\theta(0,f) = 1$ cannot be dropped in Theorem 2.

### Conclusion

Nevanlinna's second theorem in difference analogue shows that a non- periodic finite order meromorphic function cannot have many values which only appear in pairs, separated by a fixed constant. All concepts of Nevanlinna theory related to ramification have a natural difference analogue. For instance, constant functions are analogous to periodic functions, and a pole with multiplicity $n > 1$ is analogous to a line of n poles with the same multiplicity, each separated by a fixed constant. Number of results on the value distribution of finite-order meromorphic functions can be derived by combining existing proof techniques from Nevanlinna's theory together with the difference analogues.

The current Clunie types of results in Lemmas regarding difference- differential polynomials are mainly useful for the problems relating to entire or meromorphic solutions of finite order for the difference- differential equations.

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