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Compact Weighted Composition Operators on a Space of Operators

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Abstract: Let $L(E)$ be a locally convex space of all continuous linear transformations from a locally convex space E into itself, which is equipped with the topology of uniform convergence on bounded subsets of E. In this paper, we characterize compact weighted composition operators on these spaces.

Keywords: *Locally convex space; topology of uniform convergence on bounded sets; weighted composition operator.*
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Introduction

Let E be a vector space and $L(E)$ a vector space of transformations from E into itself. Let $T \in L(E)$ be such that $TA \in L(E)$ whenever $A \in L(E)$. Here $TA(x) = T(Ax)$ for all $x \in E$. A multiplication operator denoted by M_T on $L(E)$ has the form : $M_T(A) = TA$ for all $A \in L(E)$. Similarly, a composition operator C_T on $L(E)$ has the form : $C_T(A) = AT$ for all $A \in L(E)$. For $S,T \in L(E)$, we have a weighted composition operator $W_{S,T}$ on $L(E)$ given by $W_{S,T}(A) = SAT$ for all $A \in L(E)$.These operators on different function spaces have been the subject matter of various papers in recent years, see for example [4], [5], [6] and the references listed therein. For a non-trivial locally convex space E, let $\boldsymbol{P}$ denote the family of all continuous seminorms on E and $\boldsymbol{B}$ denote the family of all bounded subsets of E. For each $p \in \boldsymbol{P}$ and $K \in \boldsymbol{B}$, we define the seminorm $\| . \|_{p, K}$ on $L(E)$ as $\| . \|_{p, K} = \sup \{p(Ax) : x \in K\}$ for all $A \in L(E)$.Then the family $\{\| . \|_{p, K}: p \in \boldsymbol{P}, K \in \boldsymbol{B}\}$ of seminorms defines a locally convex by $L_b(E)$. The convergence in this topology is the uniform convergence on bounded subsets of E. For details about topologies on a space of operators, we refer to Grothendieck [2] and Kothe [3].

Operator On $L_b(E)$

Composition operators, multiplication operators and weighted composition operators on $L_b(E)$ has been characterized by Singh, Singh and Takagi [6]
whereas idempotent and invertible properties of such operators were characterized by Singh, Singh and the author in [5]. The following theorems have been proved in [6].

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Theorem A: Let T be a linear transformation on E. Then C_T is a composition operator on $L_b(E)$ if and only if T is continuous on E. Also, M_T is a multiplication operator on $L_b(E)$ if and only if T is continuous on E. Using this result, it follows that if S and T are continuous linear transformations on E, then the associated transformation $W_{S,T}$ $(= M_S\, C_T)$ is a weighted composition operator on $L_b(E)$. The converse statement is not true. For instance, take S to be the zero operator O on E. Then the transformation $W_{O,T}$ is a weighted composition operator(in fact, zero operator) on $L_b(E)$, even if T is not continuous. The case $T=O$ gives rise to the similar situation. If these two possibilities are excluded, the converse is also valid which is given as under:

Theorem B: Let E is a metrizable locally convex space and S, T be non-zero linear transformations on E. Then $W_{S,T}$ is a weighted composition operator on $L_b(E)$ if and only if both S and T are continuous. The most important examples of metrizable locally convex spaces often have the structure of normed linear spaces. Let E be a normed linear space with norm $\|\ .\ \|_E$. Then $L_b(E)$ is the usual space of all continuous (or bounded) linear transformations on E. It is an algebra with norm:

$\|T\|_L = \sup\{\|T x\|_E : x \in B_1\}$, where $B_1 = \{x \in E : \|x\|_E \leq 1\}$. Similarly, if we define $\|W_{S,T}\| = \sup\{\|W_{S,T}(A)\|_L : A \in L_b(E), \|A\|_L \leq 1\}$

Theorem C: Let E be a normed linear space, and let S and T be non zero linear transformations on E. Then $W_{S,T}$ is a weighted composition operator on $L_b(E)$ if and only if both S and T are continuous. In this case, $\|W_{S,T}\| = \|S\|_L \|T\|_L$. We present two examples of operators S and T: one where the corresponding transformation $W_{S,T}$ is a operator and in the second, it is not an operator.

Example 1: *Let* $E = l^2$ be the Hilbert space of all square summable sequences of complex numbers. For a bounded sequence $\{a_n\}$ of complex numbers, define a transformation S on l^2 as $S(\{x_n\}) = \{a_n x_n\}$ for all $\{x_n\}$ in l^2. Then $S \in L_b(E)$ and $\|S\|_L = \sup\{\,|a_n| : n \in \mathbf{N}\}$. Also, let T be the unilateral shift operator on E. Then $T \in L_b(E)$ and $\|T\|_L = 1$. Using Theorem C, it follows that $W_{S,T}$ is a weighted composition operator on $L_b(E)$ and $\|W_{S,T}\| = \|S\|_L$.

Example2: Let $E = C^\infty[0,1]$ be the linear space of all infinitely differentiable functions f in [0,1] such that the n^{th} derivative $f^{(n)}$ is continuous on [0,1] for all n. Under the supremum norm, E becomes a normed linear space but it is not complete. Now define a linear transformation T on E as $Tf = f'$ for all $f \in E$. Then T is not bounded ([1]) and so this T does not induce a composition operator C_T nor a weighted composition operator $W_{T,T}$ on $L_b(E)$.

Compact Weighted Composition Operator

Compactness of weighted composition operators on L^p-spaces, H^p-spaces and on spaces of continuous functions has been studied by various authors. For details, we refer to [4]. The aim of this paper is to study compact weighted composition operators on $L_b(E)$, when E is a Banach space with the norm $\|\,.\,\|$. For a Banach space E, $L_b(E)$ is denoted by $B(E)$ and is a Banach algebra with the norm given by

$\| A \|_B = \sup\{\| A x \| : x \in B_1\}$, where $B_1 = \{x \in E : \| x \|_E \leq 1\}$.

For presenting the main result, we require the following two propositions:

Proposition 3.1: Let E be a Banach space and K be a bounded subset of E satisfying the condition:

(∗) "For any sequence $\{x_n\}$ in K and $\varepsilon > 0$, there corresponds a subsequence $\{x_{n_i}\}$ such that $\| x_{n_i} - x_{n_j} \| < \varepsilon$ for all x_{n_i}, x_{n_j} in the sequence." Then the closure $\overline{K}$ of K is a compact subset of E.

Proof: To prove that $\overline{K}$ is compact, we show that every sequence in K has a convergent subsequence. Let $\{x_n\}$ be a sequence in K. Using (∗) with $\varepsilon = 1$, take a subsequence $\{x_m^{(1)}\}$. Inductively, we chose a k^{th} sequence $\{x_m^{(k)}\}$ from $(k\text{-}1)^{\text{th}}$ subsequence $\{x_n^{(k-1)}\}$ using (∗) with $\varepsilon = \frac{1}{k}$. Writing k^{th} term of the k^{th} subsequence $\{x_m^{(k)}\}$ by x_k, the resulting sequence $\{x_k\}$ is a subsequence of the original sequence $\{x_n\}$ and is a Cauchy sequence. Since $\overline{K}$ is complete, so $\{x_k\}$ is convergent.

Notation: Let $x \in E$ and M be a subset of E. Then the distance from x to M is denoted by $d(x, M)$ and is given by

$$d(x, M) = \inf\{\| x-y \| : y \in M\}.$$

Proposition 3.2: Let K be a bounded subset of E satisfying the property: To any $\varepsilon > 0$, there exists a finite dimensional subspace M of E such that $d(x, M) < \varepsilon$ for all $x \in M$. Then $\overline{K}$ is compact.

Proof: By Proposition 3.1, it suffices to show that K satisfies the condition (*). Let $\{x_n\}$ be a sequence in K and $\varepsilon > 0$ be given. By our assumption, there is a finite dimensional subspace M such that $d(x, M) < \frac{\varepsilon}{3}$ for all $x \in K$. For each x_n, take an element y_n of M with $\| x_n - y_n \| < \frac{\varepsilon}{3}$. Since $\{x_n\}$ is bounded, $\{y_n\}$ becomes a bounded sequence in the finite- dimensional space M and so $\{y_n\}$ has a convergent subsequence $\{y_{n_i}\}$. This implies that there exists a number i_o such that for $i, j \geq i_o$, we have

$\| y_{n_i} - y_{n_j} \| < \frac{\varepsilon}{3}$. Therefore, for $i, j \geq i_o$, we have $\| x_{n_i} - x_{n_j} \| \leq \| x_{n_i} - y_{n_i} \| + \| y_{n_i} - y_{n_j} \| + \| y_{n_j} - x_{n_j} \| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. This shows that the subsequence $\{x_{n_i} : i \geq i_o\}$ satisfies the condition (*). Hence our result follows from proposition 3.1.

We will now present a characterization of compact weighted composition operators on $B(E)$, when E is a Banach space. By B_2, we denote the closed unit ball of $B(E)$.

Theorem 3.3: Let S, T be non-zero operators on E. Then $W_{S,T}$ is a compact weighted composition operator on $B(E)$ if and only if both S and T are compact.

Proof: Suppose that S and T are compact. To prove that $W_{S,T}$ is compact, we must show that the closure of $W_{S,T}(B_2)$ is compact. Using Proposition 3.1 with $E = B(E)$ and $K = W_{S,T}(B_2)$, we see the following statement is to be proved: For any sequence $\{A_n\}$ in B_2 and $\varepsilon > 0$, there corresponds a subsequence $\{A_{n_i}\}$ such that $\| W_{S,T} A_{n_i} - W_{S,T} A_{n_j} \|_B < \varepsilon$ for all pairs A_{n_i}, A_{n_j} in the subsequence. Take ε_1 with $0 < \varepsilon_1 \| S \|_B < \frac{\varepsilon}{3}$, and put $N(x) = \{y \in E: \| y-x \| < \varepsilon_1 \}$ for $x \in T(B_1)$. Since the closure of $T(B_1)$ is compact, we can choose finitely many element $x_1, x_2, \ldots\ldots x_m$ of $T(B_1)$ such that $T(B_1)$ □ $\bigcup_{k=1}^{m} N(x_k)$. For each $k = 1, 2, 3, \ldots.., m$, define a k^{th} sequence $\{y_n^{(k)}\}$ by $\{y_n^{(k)}\} = A_n x_k$ for $n = 1, 2, \ldots..$ Then each sequence $\{y_n^{(k)}\}$ is bounded, and so by compactness of S, we can find a subsequence $\{n_i\}$ of $\{n\}$ such that m sequences $\{S\, y_{n_i}^{(1)}\}$, $\{Sy_{n_i}^{(2)}\}, \ldots\ldots, \{Sy_{n_i}^{(m)}\}$ are all convergent. Here we may assume that $\{n_i\}$ satisfies

$$\|Sy_{n_i}^{(k)} - Sy_{n_j}^{(k)}\| < \frac{\varepsilon}{3}$$

for all pairs n_i, n_j and $k = 1, 2, 3, \ldots.., m$. Now let $z \in B_1$ be arbitrary. Then by our choice of $x_1, x_2, \ldots\ldots, x_m$, there exists atleast one x_k such that $T(z) \in N(x_k)$, that is,

$$\| T(z) - x_k \| < \varepsilon_1.$$

If n_i and n_j are elements of the above subsequence $\{n_i\}$, then

$$\|W_{S,T} (A_{n_i} (z)) - W_{S,T} (A_{n_j}(z))\|$$

$$\leq \| S A_{n_i} T(z) - S A_{n_j} T(z) \|$$

$\leq \| S A_{n_i} T(z) - S A_{n_i} (x_k) \| + \| S A_{n_i} (x_k) - S A_{n_j} (x_k) \| + \| S A_{n_j} (x_k) - S A_{n_j} T(z) \|$

$\leq \| S \|_B \| A_{n_i} \|_B \| T(z) - x_k \| + \| Sy_{n_i}^{(k)} - Sy_{n_j}^{(k)} \| + \| S \|_B \| A_{n_j} \|_B \| x_k - T(z) \|$

$< \varepsilon_1 \| S \|_B + \frac{\varepsilon}{3} + \varepsilon_1 \| S \|_B < \varepsilon.$

Since $z \in B_1$ was arbitrary, we obtain $\| W_{S,T} A_{n_i} - W_{S,T} A_{n_j} \|_B < \varepsilon.$

For the other way part, suppose $W_{S,T}$ is compact. We first show that S is compact. Let $\{y_n\}$ be a sequence in B_1 and choose non-zero elements x_o and z_o of E such that $T(z_o) = x_o$. By Hahn-Banach Theorem, there is a continuous linear functional f of E such that $f(x_o) = 1$ and $\| f \| = 1$. For any n, define $A_n : E \to E$ as $A_n(x) = f(x)y_n$ for each $x \in E$. Then clearly $\{A_n\}$ is a sequence in $B(E)$ and $\|A_n\| \leq 1$, that is, it is a sequence in B_2. Since $W_{S,T}$ is compact, there exists a subsequence $\{ A_{n_i} \}$ of $\{A_n\}$ and $A \in B(E)$ such that $\| W_{S,T} A_{n_i} - A\|_B \to 0.$

Also, we have $W_{S,T}(A_n(z_o)) = S A_{n_o} T(z_o) = S A_{n_o} (x_o) = S(f(x_o)y_n) = S(y_n)$ and so $\| S y_{n_i} - A(z_o) \| = \| W_{S,T} (A_{n_i} (z_o)) - A(z_o) \| \leq \| W_{S,T} A_{n_i} - A\|_B \| z_o \| \to 0.$

Thus we find a subsequence $\{ y_{n_i} \}$ such that $\{Sy_{n_i} \}$ is convergent. Hence S is compact.

Finally, we show that T is compact. To see this, we assume on the contrary that T is not compact, that is, $\overline{T(B_1)}$ is not a compact set. Then, by Proposition 3.2, there exists a δ such that $0 < \delta \leq 1$ and has the property:

For any finite dimensional subspace M, there exits an element x of $T(B_1)$ such that $d(x, M) \geq \delta$. Fix a non-zero element x_1 of $T(B_1)$, and choose y_o so that $S(y_o) \neq 0$ and $\| y_o \| \leq \delta$. By Hahn Banach Theorem, we find an operator $A_1 \in B(E)$ such that $A_1(x_1) = y_o$ and $\| A_1 \|_B \leq \delta \ (\leq 1)$.

Next we construct an operator A_2 as follows:

If M_1 is the subspace spanned by x_1, the choice of δ gives an element x_2 in $T(B_1)$ such that $d(x_2, M_1) \geq \delta$. Using Hahn-Banach Theorem again, we find a continuous linear functional f_2 on E such that $f_2(x_1) = 0, f_2(x_2) = 1$ and $\| f_2 \| \leq \frac{1}{d(x_2, M_1)}$.

Define $A_2: E \to E$ as $A_2(x) = f_2(x)y_o$ for all $x \in E$. Then clearly $A_2 \in B(E)$ and satisfies

$$A_2(x_1) = 0,\ A_2(x_2) = y_o \text{ and } \| A_2 \|_B \leq 1.$$

Continuing in this way, we obtain a sequence $\{x_n\}$ in $T(B_1)$ and a sequence $\{A_n\}$ of operators in B_2 such that $A_n(x_m) = 0$ for $m = 1, 2, \ldots\ldots, n-1$ and $A_n(x_n) = y_o$.

Let z_n be an element of B_1 with $T(z_n) = x_n$. Then for any m, n with $m < n$, we have

$\|W_{S,T}(A_m) - W_{S,T}(A_n)\|_B \geq \|W_{S,T}(A_m(z_n)) - W_{S,T}(A_n(z_n))\| = \| SA_mT(z_n) - SA_nT(z_n) \|$

$= \| SA_m(x_n) - SA_n(x_n) \|$

$= \| 0 - S(y_o) \| = \| S(y_o) \| > 0$, which implies that $\{W_{S,T}(A_n)\}$ cannot have a convergent subsequence, while $\{A_n\}$ is a bounded sequence. This is contrary to the compactness of $W_{S,T}$. Thus we conclude that T must be compact.

Corollary3.4 Let E be an infinite dimensional Banach space. Then a composition operator C_T on $B(E)$ is compact if and only if T is a zero operator O on E. Also, a multiplication operator M_T on $B(E)$ is compact if and only if $T = O$ on E.

Proof: Let C_T(respectively, M_T) be a compact composition (respectively, multiplication) operator on $B(E)$. Then C_T(respectively, M_T) can be considered as a weighted composition operator $W_{I,T}$(respectively, $W_{T,I}$), where I is the identity operator on E. If T is non-zero, then the above Theorem 3.3 shows that I is compact. This contradicts the assumption that E is infinite dimensional. Hence T must be zero operator on E. The converse part is trivial.

Remark: The compactness of the weighted composition operator $W_{S,T}$ on $L_B(E)$ when E is a locally convex space will be worthwhile to prove and we are working in this direction.

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