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Solution of Generalized Mixed Set-Valued Variational Inequality Problem

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Abstract: In this paper, we use auxiliary principal technique to suggest a new class of predictor-corrector algorithms for solving general mixed equilibrium problems. The convergence of the proposed methods either requires partially relaxed strongly monotonicity. As special cases we obtain a number of known and new results for solving various classes of equilibrium and variational inequalities.

Keywords and phrases: *Equilibrium problem; Auxiliary principle; Iterative algorithm; Monotone; Partially relaxed strongly mixed monotone; Variational inequalities; Iterative methods; Convergence.*
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Introduction

In accordance with [1], under an equilibrium problem (EP) we understand the problem of finding $\bar{x}$ in a set K so as to satisfy $F(\bar{x}, y) \geq 0, \forall y \in K,$ where F is a real-valued bifunction on K. This was initially motivated by the earlier work of Giannessi [2] who first introduced vector variational inequalities in finite-dimensional Euclidean spaces. Many interesting and sophisticated problems in applied mathematics can be cast the form of an EP, as in the fields of optimization, mathematical economics, networks, and mechanics. For solving variational inequalities problems, Noor [9] has used the auxiliary principal technique to suggest some iterative methods. The auxiliary principal technique is mainly due to Lions and Stanpacchia [5] and Glowinski is approach to study the existence of a solution of the mixed variational inequalities. In recent years, Noor [7-9] has used this technique to study some predictor-corrector methods for classes of equilibrium and variational inequality problems. In this paper, we use the auxiliary principal technique to suggest a class of three-step predictor-corrector iterative methods for variational inequalities problems. Consequently, our results represent an improvement and refinement of the previously known results.

Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot,\cdot \rangle$ and $\| \cdot \|$, respectively. Let $C(H)$ be the family of all non-empty compact subsets of H. Let $T : H \to C(H)$ be a multivalued operator and $g : H \to H$ be a single-valued operator. Let K be a

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non-empty, closed and convex set in H .

For a given single-valued bi-function, $F(\cdot,\cdot): H \times H \to R$ we consider the problem of finding $u \in H, g(u) \in K, v \in T(u)$ such that $F(v, g(v)) + \langle v, g(v) - g(u) \rangle \geq 0, \ \forall\, g(v) \in K$ (2.1)

The inequality of type (2.1) is called the *Generalized Mixed Set-Valued Variational Inequality Problem* . It can be shown that a wide class of multivalued free, obstacle, moving, equilibrium and optimization problems arising in various branches of pure and vapplied sciences can be studied in the general framework of generalized mixed set-valued variational inequality problem, See [1-10] and the references therein.

We also need the following well known results and concepts.

Lemma.2.1. For all $u, v \in H$, we have

$2\langle u, v \rangle = \| u + v \|^2 - \| u \|^2 - \| v \|^2$ (2.2)

Definition 2.1. For all $u_1, u_2, z \in K, w_1 \in T(u_1), w_2 \in T(u_2)$ the multivalued operator $T: H \to C(H)$ is said to be :

1. g -partially relaxed strongly monotone, if there exists a constant $\alpha > 0$, $a > 0$ such that

$$F(w_1, g(u_2)) + F(w_2, g(z)) \leq \alpha \| g(z) - g(u_1) \|^2;$$

and

$$\langle w_1 - w_2, g(u_2) - g(z) \rangle \leq a \| g(z) - g(u_1) \|^2;$$

2. g - co-coercive, if there exists a constant $\mu > 0$ such that

$$\langle w_1 - w_2, g(u_1) - g(u_2) \rangle \geq \mu \| w_1 - w_2 \|^2;$$

3. g - monotone iff

$$F(w_1, g(u_2)) + F(w_2, g(u_1)) \leq 0.$$

and

$$\langle w_1 - w_2, g(u_1) - g(u_2) \rangle \geq 0$$

Definition 2.2. For all $u_1,u_2 \in H, w_1 \in T(u_1), w_2 \in T(u_2)$, the multivalued operator $T: H \to C(H)$ is said to be $M-Lipschitz\ \ continuous$ iff there exists a constant $\delta > 0$, such that

$M(T(u_1),T(u_2)) \leq \delta \| \ u_1 - u_2 \ \|$ where $M(\cdot,\cdot)$ is the Hausdorff metric on $C(H)$.

We remark that if $z = u_1$, then $g-$ partially relaxed strong monotonicity is exactly g - monotonicity of the operator T. It has been shown [11] that $g-$ co-coerseivity implies g-partially relaxed strongly monotonicity, but the converse is not true. This shows that concept of g-partially relaxed strongly monotonicity is weaker than g-co-coercivity. For the single-valued operator T, Definition 2.1 reduces to the definition of partially relaxed strongly monotonicity of the operator.

Main results

In this section, we suggest and analyze a new iterative method for solving the problem (2.1) by using the auxiliary principle technique.

For a given $u \in H, T(u), g(u) \in K, v \in T(u)$ consider the problem of finding a solution $w \in H, g(w) \in K$, satisfying the auxiliary generalized mixed equilibrium problem (2.1)

$$\rho F(v, g(v)) + \langle \rho v + g(w) - g(u), g(v) - g(w) \rangle \geq 0, \text{ for all } g(v) \in K (3.1)$$

where $\rho > 0$ is a constant. We note that, if $w = u$, then clearly w is a solution of the generalized mixed set-valued variational inequality problem (2.1). This observation enable us to suggest the following predictor-corrector method for solving the generalized mixed set-valued variational inequality problem (2.1).

Algorithm 3.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\rho F(\eta_n, g(v)) + \langle \rho\eta_n + g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1}) \rangle \geq 0, \forall g(v) \in K (3.2)$$

$$\eta_n \in T(w_n) : |\eta_{n+1} - \eta_n| \leq M(T(w_{n+1}), T(w_n)), (3.3)$$

$$\beta F(\xi_n, g(v)) + \langle \beta\xi_n + g(w_n) - g(y_n), g(v) - g(w_n) \rangle \geq 0, \forall g(v) \in K (3.4)$$

$$\xi_n \in T(y_n) : \| \ \xi_{n+1} - \xi_n \ \| \leq M(T(y_{n+1}), T(y_n)), (3.5)$$

$$\mu F(\gamma_n, g(v)) + \langle \mu\gamma_n + g(y_n) - g(u_n), g(v) - g(y_n) \rangle \geq 0, \forall g(v) \in K, (3.6)$$

$$\gamma_n \in T(y_n): |\gamma_{n+1} - \gamma_n| \leq M(T(u_{n+1}), T(u_n)), (3.7)$$

Where $n=0,1,2,..$ and $\rho>0, \beta>0$ and $\mu>0$ are constants.

Note that, if $F(v, g(v)) = \langle v, g(v) - g(u) \rangle$,then Algorithm 3.1 reduces to the following iterative method.

Algorithm 3.2. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes

$$\langle \rho\eta_n + g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1}) \rangle \geq 0, \forall g(v) \in K$$

$$\eta_n \in T(w_n): |\eta_{n+1} - \eta_n| \leq M(T(w_{n+1}), T(w_n)),$$

$$\langle \beta\xi_n + g(w_n) - g(y_n), g(v) - g(w_n) \rangle \geq 0, \forall g(v) \in K$$

$$\xi_n \in T(y_n): |\xi_{n+1} - \xi_n| \leq M(T(y_{n+1}), T(y_n)),$$

$$\langle \mu\gamma_n + g(y_n) - g(u_n), g(v) - g(y_n) \rangle \geq 0, \forall g(v) \in K;$$

$$\gamma_n \in T(y_n): |\gamma_{n+1} - \gamma_n| \leq M(T(u_{n+1}), T(u_n)),$$

where $n=0,1,2,..$ and $\rho>0, \beta>0$ and $\mu>0$ are constants.

which is called the predictor-corrector method for solving generalized variational inequalities.

For a suitable choice of the operators and the space H, one can obtain various new and known methods for solving equilibrium and variational inequality problems.

Theorem 3.1. Let H be a finite dimensional space. Let $T: H \to C(H)$ g - partially relaxed strongly monotone operators with constant $\alpha, a > 0$. Let $F(\cdot,\cdot): H \times H \to R$ be a bi-functional and $g: H \to H$ be invertible and $0 < \rho < \dfrac{1}{2(\alpha + a)}$. If u_{n+1} is the approximate solution obtained from Algorithm 3.1 and $u \in H$ is the exact solution of (2.1), then $lim_{n\to\infty} u_n = u$.

For the proof of the theorem we need the following result.

Lemma: 3.1. Let $u \in H$ be the exact solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 3.1. Let $F(\cdot,\cdot): H \times H \to R$ be a bi-functional and $g: H \to H$ be invertible and

$0 < \rho < \dfrac{1}{2(\alpha + a)}$. If $T : H \to C(H)$, g-partially relaxed strongly monotone operators with constant $\alpha > 0$, then

$$\| \, g(u_{n+1}) - g(u) \, \|^2 \le \| \, g(u_n) - g(u) \, \|^2 - (1 - 2\rho(\alpha + a))\| \, g(u_{n+1}) - g(u_n) \, \|^2 \quad (3.8)$$

where $0 < \rho < \dfrac{1}{2(\alpha + a)}$

Proof:- Let $u \in H$, be a solution of (2.1). Then $\rho F(v, g(v)) + \langle \rho v, g(v) - g(u) \rangle \ge 0, \forall \, g(v) \in K$(3.9)

$\beta F(v, g(v)) + \langle \beta v, g(v) - g(u) \rangle \ge 0, \forall \, g(v) \in K$(3.10)

$\mu F(v, g(v)) + \langle \mu v, g(v) - g(u) \rangle \ge 0, \forall \, g(v) \in K$(3.11) where $\rho > 0, \beta > 0, \mu > 0$ are constants.

Now taking $v = u_{n+1}$ in (3.9) and $v = u$ in (3.2), we have

$\rho F(v, g(u_{n+1})) + \langle \rho v, g(u_{n+1}) - g(u) \rangle \ge 0$,(3.12) and

$\rho F(\eta_n, g(u)) + \langle \rho \eta_n + g(u_{n+1}) - g(w_n), g(u) - g(u_{n+1}) \rangle \ge 0$
(3.13)

Adding (3.12) and (3.13), we have

$$\begin{aligned} \langle g(u_{n+1}) - g(w_n), g(u) - g(u_{n+1}) \rangle &\ge - \rho[F(\eta_n, g(u)) + F(v, g(u_{n+1})) + \langle \eta_n - v, g(u) - g(u_{n+1}) \rangle] \\ &\ge - (\alpha + a)\rho \| \, g(u_{n+1}) - g(w_n) \, \|^2 \end{aligned} \quad (3.14)$$

Where we have used the fact that T is g-partially relaxed strongly monotone with constant $\alpha, a > 0$.

Setting $u = g(u) - g(u_{n+1})$ and $v = g(u_{n+1}) - g(w_n)$ in (2.2), we obtained

$$\langle g(u_{n+1}) - g(w_n), g(u) - g(u_{n+1}) \rangle = \frac{1}{2}\{ \| \, g(u) - g(w_n) \, \|^2 - \| \, g(u_{n+1}) - g(w_n) \, \|^2 - \| \, g(u) - g(u_{n+1}) \, \|^2 \} \quad (3.15)$$

Combining (3.14) and (3.15), we have

$$\| g(u_{n+1})-g(u) \|^2 \le \| g(w_n)-g(u) \|^2 -(1-2\rho(\alpha+a))\| g(u_{n+1})-g(w_n) \|^2 \quad (3.16)$$

Taking $v=u$ in (3.4) and $v=w_n$ in (3.10), we have $\beta F(v,g(w_n))+\langle \beta v, g(w_n)-g(u)\rangle \ge 0, (3.17)$

and $\beta F(\xi_n,g(u))+\langle \beta\xi_n+g(w_n)-g(y_n), g(u)-g(w_n)\rangle \ge 0, (3.18)$

Adding (3.17) and (3.18), we have

$$\begin{aligned}\langle g(w_n)-g(y_n), g(u)-g(w_n)\rangle &\ge -\beta\ [F(\xi_n,g(u))+F(v,g(u))+\langle \xi_n-v, g(u)-g(w_n)\rangle] \\ &\ge -\beta(\alpha+a)\| g(y_n)-g(w_n) \|^2 \quad (3.19)\end{aligned}$$

Since T is g-partially relaxed strongly monotone operators with constant $\alpha, a>0$.

Now taking $u=g(u)-g(w_n)$ and $v=g(w_n)-g(y_n)$ in (2.2), we have

$$\begin{aligned}&\langle g(w_n)-g(y_n), g(u)-g(w_n)\rangle \\ &= \frac{1}{2}\{\| g(u)-g(y_n) \|^2 - \| g(w_n)-g(y_n) \| - \| g(u)-g(w_n) \|^2\} \quad (3.20)\end{aligned}$$

Combining (3.19) and (3.20), we have

$$\begin{aligned}\| g(u)-g(w_n) \|^2 &\le \| g(u)-g(y_n) \|^2 -(1-2\beta(\alpha+a))\| g(y_n)-g(w_n) \|^2 \\ &\le \| g(u)-g(y_n) \|^2, \text{for } 0<\beta<\frac{1}{2(\alpha+a)} \quad (3.21)\end{aligned}$$

Similarly, by taking $v=u$ in (3.6) and $v=y_n$ in (3.11) and using the g-partially relaxed strongly monotonicity of the operator T , we have

$$\langle g(y_n)-g(u_n), g(u)-g(y_n)\rangle \ge -\mu(\alpha+a)\| g(y_n)-g(u_n) \|^2 \quad (3.22)$$

Letting $u=g(u)-g(y_n)$ and $v=g(y_n)-g(u_n)$ in (2.2), and combining the resultant with (3.22), we have

$$\begin{aligned}\| g(y_n)-g(u) \|^2 &\le \| g(u)-g(u_n) \|^2 -(1-2\mu(\alpha+a))\| g(y_n)-g(u_n) \|^2 \\ &\le \| g(u)-g(u_n) \|^2, \quad \text{for } 0<\mu<\frac{1}{2(\alpha+a)} \quad (3.23)\end{aligned}$$

$$\begin{aligned}\| g(u_{n+1})-g(w_n) \|^2 &= \| g(u_{n+1})-g(u_n)+g(u_n)-g(w_n) \|^2 \\ &= \| g(u_{n+1})-g(u_n) \|^2+\| g(u_n)-g(w_n) \|^2 \\ &\quad +2\langle g(u_{n+1})-g(u_n), g(u_n)-g(w_n)\rangle \qquad (3.24)\end{aligned}$$

Combining (3.16), (2.21), (3.23)and (3.24), we obtain

$$\| g(u_{n+1})-g(u) \|^2 \le \| g(u_n)-g(u) \|^2 -(1-2\rho(\alpha+a))\| g(u_{n+1})-g(u_n) \|^2$$

where $0<\rho<\frac{1}{2(\alpha+a)}$, the required result (3.5)

Proof of the Theorem 3.1 : Let $u\in H$ be a solution of (2.1). Since $0<\rho<\frac{1}{2(\alpha+a)}$, from (3.8), it follows that the sequence $\{\| g(u)-g(u_n) \|\}$ is non increasing and consequently $\{u_n\}$ is bounded. Furthermore, we have $\sum_{n=0}^{\infty}(1-2(\alpha+a)\rho)\| g(u_{n+1})-g(u_n) \|^2 \le \| g(u_0)-g(u_n) \|^2$,

which implies that $\lim_{n\to\infty}\| g(u_{n+1})-g(u_n) \|=0 (3.25)$. Let $\hat{u}$ be the cluster point of $\{u_n\}$ and let the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converges to $\hat{u}\in H$. Replacing w_n and y_n by $\{u_{n_j}\}$ in (3.2), (3.4) and (3.6), taking the limit $n_j\to\infty$ and using (3.25), we have $F(\hat{v},g(v))+\langle\hat{v},g(v)-g(\hat{u})\rangle\ge 0,\ \forall\, g(v)\in K,$ which implies that $\hat{u}$ solves the generalized mixed set-valued variational inequality problem (2.1) and $\| g(u_{n+1})-g(u) \|^2 \le \| g(u_n)-g(u) \|^2$ Thus, it follows from the above inequality that $\{u_n\}_1^{\infty}$ has exactly one limit point $\hat{u}$ and $\lim_{n\to\infty} g(u_n)=g(\hat{u})$ Since g is invertible, thus $\lim_{n\to\infty} u_n=\hat{u}.$ It remains to show that $v\in T(u)$. From (3.7) and using the M-Lipschitz continuity of T, we have $\| v_n-v \|\le M(T(u_n),T(u))\le\delta\| u_n-u \|,$ which implies that $v_n\to v$ as $n\to\infty$. Now consider

$$\begin{aligned}d(v,T(u)) &\le \| v-v_n \|+d(v,T(u)) \\ &\le \| v-v_n \|+M(T(u_n),T(u)) \\ &\le \| v-v_n \|+\| u_n-u \|\to 0. \\ &\quad \text{as } n\to\infty\end{aligned}$$

Where $d(v,T(u)) = \inf\{\| v-z \| : z \in T(u)\}$, and $\delta > 0$ is the M-Lipschitz continuity constant. From the above inequality, it follows that $d(v,T(u)) = 0$. This implies that $v \in T(u)$, since $T(u) \subseteq C(H)$.

This complete the proof. $[.1in]$

If $F(v,g(v)) = \langle v, g(v) - g(u) \rangle$, then Algorithm 3.2 a predictor-corrector method for solving the following variational inequalities.

Corollary 3.1. Let H be a finite dimensional space.Let $T : H \to C(H)$ is g- partially relaxed strongly monotone operators with constant $\alpha, a > 0$. Let $g : H \to H$ be invertible and $0 < \rho < \dfrac{1}{2(\alpha + a)}$. If u_{n+1} is the approximate solution obtained from Algorithm 3.2 and $u \in H$ is the exact solution of

$\langle v, g(v) - g(u) \rangle \geq 0, \ \forall \, g(v) \in K (3.26)$

then $\lim_{n \to \infty} u_n = u$.

Corollary 3.2. If $g = I$, the identity operator , then corollary 3.1 is equivalent to find $u \in K$ such that $\langle v, v - u \rangle \geq 0, \ \forall \, v \in K$ then $\lim_{n \to \infty} u_n = u$.

Remark

The corollary 3.1 is called multivalued variational inequality. It is known that a wide class of multivalued odd order and non symmetric free, obstacle moving, equilibrium and optimization problems arising in pure and applied sciences can be studied via the multivalued variational inequalities (3.26); see, for example, Noor[7,8] The corollary 3.2.is called the generalized variational inequalities introduced and studied by Fang and Pererson[5] see also [5,9] and reference therein.

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