Vol. (1) No. (1) March 2018

JK Research Journal in Mathematics and Computer Sciences

# Composite Volterra Integral Operators on Orlicz Spaces

*Nidhi Suri[1] B.S. Komal[2]

[1]*Govt. P.G. College for Women, Gandhi Nagar, Jammu, J&K, India*
[2]*Department of Engineering Mathematics, M.I.E.T., Kot Bhalwal, J&K, India*

**Abstract:**In this paper we explore a condition for the bounded Composite Volterra integral operator. The adjoint of the composite Volterra integral operator is also obtained in this paper.

**Keywords:** *Bounded Operator; Composite Volterra integral Operator ;Expectation operator ; Radon Nikodym derivative; Volterra integral Operator.* ***Mathematics Subject Classification:*** *Primary 47B38 ; Secondary 47B99*

## Introduction

A convex function $\Phi\colon \mathrm{R} \to \mathrm{R}^+$ is called a young function if it satisfies the following properties:

(i) $\Phi(x) = \Phi(-x)$ for every $x \in \mathrm{R}$ ,

(ii) $\Phi(0) = 0$ ,

(iii) $\lim_{x\to\infty} \Phi(x) = \infty$

With each young function $\Phi$ we can associate another young function $\psi\colon \mathrm{R} \to \mathrm{R}^+$ defined by $\psi(y) = sup\{x|y| - \Phi(x) : x \geq 0\}$ for each $y \in R$.The function $\Psi$ is called the complementary function of $\Phi$. Let $(X, s, \mu)$ be a $\sigma$-finite measure space and $X \subset R$. Define $L_\Phi(\mu) = \left\{f/f : X \to R \; is \; measurable \; function \; and \; \int_X \Phi(\alpha|f|)d\mu < \infty \; for \; some \; \alpha > 0\right\}$

For $f \epsilon L_\Phi(\mu)$, we define

$$\|f\|_\Phi = inf\left\{\epsilon > 0 : \int_X \Phi\left(\frac{|f|}{\epsilon}\right) d\mu \leq 1\right\}.$$

Then $L_\Phi(\mu)$ is a Banach space under the norm $\|.\|_\Phi$. If $\Phi(x) = |x|^p$ for every $x \in R$, then $L_\Phi(\mu) = L_p(\mu)$, the well know Banach space of $p^{th}$ integrable functions defined on $X$.

For $X = N$, the set of natural numbers and $\mu$ be the counting measure on $P(N)$ ,the power set of $N$, $L_\Phi(X) = l_\Phi(N) = \left\{f/f : N \to C \; and \; \sum_{n=1}^{\infty} \Phi\left(\frac{|f_n|}{\epsilon}\right) < \infty \; for \; some \; \epsilon > 0\right\}$

The space $l_\Phi(N)$ is known as Orlicz sequence space.The space of all bounded linear functionals on $L_\Phi(\mu)$ is denoted by $L^*_\Phi(\mu)$A measurable transformation $T\colon (X, s) \to (X, s)$ is

drnidhisuri@gmail.com
*Nidhi Suri

called non singular if $\mu(T^{-1}(E)) = 0$, whenever $\mu(E) = 0$

for each measurable subset $E$ $of$ $X$ . If $T$ is non–singular, then the measure $\mu T^{-1}$ is absolutely continuous with respect to the measure $\mu$. Therefore by the Radon Nikodym theorem, there exists a positive measurable function $f_0$ such that $\mu(T^{-1}(E)) = \int_E f_0 d\mu$. The function $f_0$ is called the Radon Nikodym derivative of the measure $\mu T^{-1}$ with respect to the measure $\mu$. The Conditional Expectation operator $E$ with respect to $T^{-1}(s)$ is a bounded projection from $L_p(X, s, \mu)$ into $L_p(X, T^{-1}(s), \mu)$. The operator $E$ has the following properties:

1. $E(f.goT) = E(f).(goT)$
2. If $f \geq g$ almost everywhere, then `$E(f) \geq E(g)$ almost everywhere.
3. $E(f)$ has the form $E(f) = goT$ for exactly one $T^{-1}(s)$ measurable function $g$.

In particular, $g = [E(f)]oT^{-1}$ is well defined measurable function.

For $X = [0,1]$, the Volterra integral operator

$$V: L_\Phi(\mu) \to L_\Phi(\mu)$$

is defined by

$$(Vf)(x) = \int_0^x f(y)\, d\mu(y)$$

The Composite Volterra integral operator

$V_T: L_\Phi(\mu) \to L_\Phi(\mu)$

is defined by $(V_T f)(x) = \int_0^x f(T(y)) d\mu(y)$The Volterra integral operator on $L_p(\mu)$ has received great attention of several mathematicians in recent years. For literature concerning Orlicz spaces,integral operators we refer to Gupta and Komal[4], Suri and Komal[6], Whitley[8]

**Bounded Composite Volterra Integral Operators**

In this section we study bounded composite Volterra integral operators

**Theorem 2.1:** Let T: [0,1] $\rightarrow$[0,1] be a measurable transformation and $\mu$ be the probability measure. Then Volterra Integral Operator $V_T : L_\Phi(\mu) \rightarrow L_\Phi(\mu)$ is a bounded operator if $f_0$ is an essentially bounded function.

Proof: Let f $\in L_\Phi(\mu)$. Choose M > 1 such that ess.sup $f_0 = \|f_0\|_\infty$<M.

Then

$$\int_0^1 \Phi\left(\frac{V_T f(x)}{M\|f\|}\right) d\mu(x) = \int_0^1 \Phi\left(\frac{\int_0^x f(T(y))d\mu(y)}{M\|f\|}\right) d\mu(x) \leq \int_0^1\int_0^x \Phi\left(\frac{f(T(y))}{M\|f\|}\right) d\mu(y)d\mu(x)$$

$$(\because\ \mu\ is\ a\ probability\ measure)$$

$$= \int_0^1\left(\int_{x=y}^1 d\mu(x)\right)\Phi\left(\frac{f(T(y))}{M\|f\|}\right)d\mu(y) \qquad (on\ interchanging\ limits) = \int_0^1 (1-y)\Phi\left(\frac{f(T(y))}{M\|f\|}\right)d\mu(y)$$

$$\leq \int_0^1 \Phi\left(\frac{f(T(y))}{M\|f\|}\right)d\mu(y) = \int_0^1 \Phi\left(\frac{f(y)}{M\|f\|}\right)d\mu T^{-1}(y) = \int_0^1 \Phi\left(\frac{f(y)}{M\|f\|}\right)f_0 d\mu(y) \leq \int_0^1 \frac{f_0}{\mathrm{M}}\Phi\left(\frac{f(y)}{\|f\|}\right)d\mu(y)$$

$$\leq \int_0^1 \Phi\left(\frac{f(y)}{\|f\|}\right)d\mu(y) \leq 1$$

Therefore $\|V_T f\| \leq M\|f\|$ for every $f \in L_\Phi(\mu)$ Hence $V_T$ is a bounded operator.

**Example 2.2:** Let $T$ $[0,1] \longrightarrow [0,1]$ be defined by $T(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{2} \\ 1-x, & \frac{1}{2} < x \leq 1 \end{cases}$ Then $f_0(x) \leq 1$ for all $x \in [0,1]$ which is a bounded function. Hence $V_T$ is a bounded composite volterra integral operator.

**Example 2.3:** Let T:$[0,1] \rightarrow [0,1]$ be defined by $T(x) = \frac{2}{3}x$ for $0 \leq x \leq 1$then $f_0(x) = \frac{3}{2}$ for $0 \leq x \leq \frac{2}{3} = 0$ for $\frac{2}{3} < x \leq 1$

which is a bounded function. Hence $V_T$ is a bounded composite volterra integral operator.

**Example 2.4:** Let $T$: $[0,1] \longrightarrow [0,1]$ be defined by $T(x) = x^2$.

Then $T^{-1}(x) = \sqrt{x}\ \forall x \in [0,1]$.

Now for any $M > 0$, and $f(x) = sin\,x \in L_\Phi[0,1]$,

we have

$$\int_0^1 \Phi\left(\frac{(V_T f)(x)}{M\|f\|}\right)d\mu(x) = \int_0^1 \Phi\left(\int_0^x \frac{f(T(y))d\mu(y)}{M\|f\|}\right)d\mu(x) = \int_0^1 \Phi\left(\int_0^x \frac{f(y^2)d(y)}{M\|f\|}\right)d(x)$$

$$But \int_0^x \frac{f(y^2)}{M\|f\|}d(y) = \frac{1}{M\|f\|}\int_0^x sin(y^2)dy\ Put\ y^2 = t\ or\ y = \sqrt{t} => dy = \frac{1}{2\sqrt{t}}dt$$

Therefore, $\frac{1}{M\|f\|}\int_0^x sin(y^2)dy = \frac{1}{M\|f\|}\int_0^{x^2} sin(t).\frac{1}{2\sqrt{t}}dt = \infty \forall x > 0$

Hence $V_T$ is not a bounded operator.

**Adjoint of a Composite Volterra Integral Operator**

In this section we shall compute the adjoint of Composite Volterra Integral Operator.

We know that the adjoint of a Volterra Integral operator $V: L_2[0,1] \to L_2[0,1]$ is defined as

$$(V^*g)(y) = \int_y^1 g(x)d\mu(x)$$

Every $g \in L_\Phi(\mu)$ gives rise to a bounded linear functional $F_g: L_\Phi^*(\mu) \to L_\Phi^*(\mu)$ which is defined as $F_g(f) = \int_X f(x)g(x)d\mu(x)$. We shall often identify $F_g$ with $g$.

**Theorem 3.1** Let $T: [0,1] \to [0,1]$ be a measurable transformation. Then $V_T^*g = E(V^*g)oT^{-1}. f_0$ for every $g \in L_\Phi[0,1]$

Proof: For $f \in L_\Phi[0,1]$ consider $\left(V_T^*F_g\right)(f) = F_g(V_T f)$

$$= \int (V_T f)(x)g(x)d\mu(x) = \int_0^1 \left(\int_0^x f\big(T(y)\big)d\mu(y)\right) g(x)d\mu(x) = \int_0^1 \left(\int_{x=y}^1 g(x)d\mu(x)\right) f\big(T(y)d\mu(y)\big)$$

$$= \int_0^1 (V^*g)(y)f\big(T(y)d\mu(y)\big) = \int_0^1 E(V^*g)oT^{-1}(y)f_0(y)f(y)d\mu(y) = F_{E\left((V^*g)oT^{-1}\right)(y)f_0(y)}(f)$$

Hence $V_T^*g = E(V^*g)oT^{-1}. f_0$

**Example 3.2** Let T : $[0,1] \to [0,1]$ be defined by $T(x) = 1 - x$.

*Then* $T^{-1}(x) = 1 - x$ *and so* $T = T^{-1}$ *and* $f_0(x) = \dfrac{dT^{-1}(x)}{dx} = \dfrac{d}{dx}(1 - x) = -1$

*For any* $g \in L_\Phi[0,1]$ *and* $f \in L_\phi[0,1]$

*We have* $\left(V_T^*F_g\right)(f) = F_g(V_T f)$

$= \int_0^1 g(x)\,(V_T f)(x)d\mu(x) = \int_0^1 g(x)\left(\int_0^x (foT)(y)d\mu(y)\right) d\mu(x) = \int_0^1 g(x)\left(\int_0^1 \chi_{[0,x]}(y)f\big(T(y)\big)\,d\mu(y)\right) d\mu(x) =$ $\int_0^1 g(x)\left(\int_0^1 (\chi_{[0,x]}oToT)(y)(foT)(y)d\mu\,(y)\right) d\mu(x)$ $\quad \because ToT = I, \textit{the identity function}$

$= \int_0^1 g(x)\left(\int_0^1 (\chi_{[0,x]}oT)(y)\,f(y)d\mu T^{-1}(y)\right) d\mu(x) = \int_0^1 g(x)\left(\int_0^1 (\chi_{[0,x]}oT)(y)f(y)f_0(y)d\mu\,(y)\right) d\mu(x) =$ $\int_0^1 g(x)\left(\int_0^1 \chi_{T^{-1}[0,x]}(y)f(y)f_0(y)d\mu(y)\right) d\mu(x)$ $\quad \because \chi_E oT = \chi_{T^{-1}(E)} = \int_0^1 f(y)\left(\int_0^{1-y} g(x)d\mu(x)\right) d\mu(y) =$ $\int_0^1 f(y)\,(V_g)(1-y)d\mu(y) = \int_0^1 f(y)\,V_g\big(T(y)\big)\,d\mu(y) = F_{(V_g)oT}(f)$ *for every* $f \epsilon L_\phi(\mu)$*Hence* $V_T^*F_g =$ $F_{V_g oT}$ Consequently $V_T^*g = \left(V_g\right)oT$ by identification

## References


[1] Cowen, C.C. and Maccluer, B.D. Composition operators on spaces of analytic functions , studies in advanced Mathematics, CRC Press New York(1995)

[2] Gupta, A. and Komal, B.S. Generalized integral operators on function spaces, Int. Journal of Math Analysis, Vol 3, 26(2009), 1277-1282.

[3] Gupta, A. and Komal, B.S. Weighted composite integral operators, Int. Journal of Math Analysis, Vol 3, 26(2009), 1283-1293.

[4] Gupta, A. and Komal, B.S. Volterra composition operators, Int. J. Contemp. Math. Sciences.2011; 6(7):345-351.

[5] Gupta, A. and Komal, B.S. Bounded composite integral operators. Investigations in Mathematical Sciences. 2011; 1:33-39.

[6] Suri, N. and Komal, B.S. Bounded Integro composition operators on Orlicz spaces, Romanian Journal of Mathematics and computer science, 2012, vol.2, p.73-80.

[7] Singh, R.K. and Komal, B.S. Composition operator on lp and its adjoint, Proc. Amer. Math. Soc. 70, (1978), 21-25.

[8] Whitley, R. The spectrum of a Volterra composition operator, Integral equations and operator theory Vol. 10 (1997).