Vol. (1) No. (1) March 2018

JK Research Journal in Mathematics and Computer Sciences

$\mathbb{D}$-valued 2-inner product on $\mathbb{D}$-Modules

Kulbir Singh

Department of Mathematics, Govt. Degree College Kathua, J&K - India.

Abstract: In this paper we introduce the notion of $\mathbb{D}$-valued 2-inner product on hyperbolic-valued or $\mathbb{D}$-valued modules. Further, we show that the $\mathbb{D}$-valued 2-inner product on a $\mathbb{D}$-module induces a real 2-inner product on its idempotent components. We also establish a relation between the $\mathbb{D}$-valued 2-inner product on $\mathbb{D}$-modules and the $\mathbb{D}$-valued inner product on $\mathbb{D}$-modules and real 2-inner product on real linear spaces.

Keywords: *Hyperbolic modules, hyperbolic-valued norm, real-valued 2-norm, real-valued 2-inner product, 2-inner product real linear-spaces.* ***AMS Subject Classification*** *46A22, 46A70, 46C99.*

Introduction

The notion of 2-normed real linear spaces was initially introduced by S. Gahler [7]. In 1963, Gahler introduced the concept of 2-metric spaces and later he extended his idea to 2-normed real linear spacess. Since then, many researchers have studied these spaces from different points of view and obtained various results, see, for instance [2, 3, 4, 6, 9, 15]. The notion of 2-normed spaces is basically a two dimensional analogue of a normed space which got more attention after the publication of a paper [15]. In this paper, A. White defined and investigated the concept of bounded linear 2-functionals on 2-normed real linear spaces.

Further, he proved a Hahn-Banach type extension theorem for linear 2-functionals on 2-normed real linear spaces. In [4] and [5], Diminnie, Gahler and white introduced the concept of 2-inner product spaces and gave some new characterizations of 2-inner product spaces. Till 2000, the theory of 2-norm was restricted only to real linear spaces but in 2001, S. N. Lal et al. published a paper [9] in which they introduced the concept of complex 2-normed linear spaces and established a Hahn-Banach extension theorem for complex linear 2-functionals. In [14], the authors inroduced the notion of 2-normed $\mathbb{D}$-modules over the commutating non-division ring $\mathbb{D}$ of hyperbolic numbers and proved the Hahn-Banach theorem for $\mathbb{D}$-linear 2-functionals.

In the present paper, we introduce the notion of $\mathbb{D}$-valued 2-inner product on $\mathbb{D}$-modules and further, establish its relation with the $\mathbb{D}$-valued inner product on $\mathbb{D}$-modules and real 2-inner product on real linear spaces.

singhkulbir03@yahoo.com
*Kulbir Singh

A Review of Hyperbolic Numbers

In this section we summarize some basic properties of hyperbolic numbers which can be found in more details in [1, 12, 13] and the references therein. The hyperbolic number can be seen as a particular case of bicomplex number. The ring of bicomplex numbers is the commutative ring BC defined as follows:

$\mathsf{BC} = \{Z = z_1 + jz_2 \mid z_1, z_2 \in \mathsf{C}(i)\}$ where i and j are commutating imaginary units with $i^2 = j^2 = -1$. In particular, if we put $z_1 = x_1$, $z_2 = iy_2$ with $x_1, y_2 \in \mathsf{R}$ and $k = ij$, then $Z = x_1 + ky_2$ is an element of the set D of hyperbolic numbers. Thus, the ring D of hyperbolic numbers is the commutative ring defined as $\mathsf{D} = \{a + kb \mid a, b \in \mathsf{R}, k^2 = 1 \text{ with } k \notin \mathsf{R}\}$. Let $z = a + kb \in \mathsf{D}$. Then the $\dagger$-conjugation on z is given by $z^\dagger = a - kb$. This $\dagger$-conjugation on D is an additive, involutive and multiplicative in nature. A hyperbolic number $z = a + kb$ is said to be an invertible if $zz^\dagger = a^2 - b^2 \neq 0$. Thus, inverse of $z \in \mathsf{D}$ is given by

$$z^{-1} = \frac{z^\dagger}{zz^\dagger}.$$

If both a and b are non-zero but $a^2 - b^2 = 0$, then z is a zero-divisior in D. We denote the set of all zero-divisiors in D by NC_D, that is, $\mathsf{NC}_\mathsf{D} = \{z = a + kb \mid z \neq 0, zz^\dagger = a^2 - b^2 = 0\}$.

The ring D of hyperbolic numbers is not a division ring as one can see that if $e_1 = \frac{1}{2}(1+k)$ and its $\dagger$-conjugate $e_2 = e_1^\dagger = \frac{1}{2}(1-k)$, then $e_1.e_2 = 0$, i.e., e_1 and e_2 are zero-divisiors in the ring D. The numbers e_1 and e_2 are mutually complementary idempotent components. They make up the so called idempotent basis of hyperbolic numbers. Thus, every hyperbolic number $z = a + kb$ in D can be written as : $z = e_1\alpha_1 + e_2\alpha_2$, (0.1)

where $\alpha_1 = a + b$ and $\alpha_2 = a - b$ are real numbers. Formula (0.1) is called the idempotent representation of a hyperbolic number. Further, the two sets $e_1\mathsf{D}$ and $e_2\mathsf{D}$ are (principal) ideals in the ring D such that $e_1\mathsf{D} \cap e_2\mathsf{D} = \{0\}$ and $e_1\mathsf{D} + e_2\mathsf{D} = \mathsf{D}$. Hence, we can write $\mathsf{D} = e_1\mathsf{D} + e_2\mathsf{D}$. (0.2). Formula (0.2) is called the idempotent decomposition of D. Thus the algebraic operations of addition, multiplication, taking of inverse, etc. can be realized component-wise. The set of non-negative hyperbolic numbers is given by (see [1, P. 19]), $\mathsf{D}^+ = \{z = e_1\alpha_1 + e_2\alpha_2 \mid \alpha_1, \alpha_2 \geq 0\}$. Further, for any $z,u \in \mathbb{D}$, we write $z \leq' u$ whenever $u - z \in \mathbb{D}^+$ and it defines a partial order on $\mathbb{D}$. Also, if we take $z,u \in \mathbb{R}$, then $z \leq' u$ if and only if $z \leq u$. Thus $\leq'$ is an extension of the total order $\leq$ on $\mathbb{R}$. For any $z = e_1\alpha_1 + e_2\alpha_2 \in \mathbb{D}$, the hyperbolic-valued modulus on $\mathbb{D}$ is given by

$$|z|_k = |e_1\alpha_1 + e_2\alpha_2|_k = e_1|\alpha_1| + e_2|\alpha_2| \in \mathbb{D}^+, \qquad (0.3)$$

where $|\alpha_1|$ and $|\alpha_2|$ denote the usual modulus of real numbers α_1 and α_2 respectively. For more details, see ([1, Section 1.5], [12] and [13]).

Let X be a $\mathbb{D}$-module. Consider the sets $X_1 = e_1X$ and $X_2 = e_2X$. Then

$$X_1 \cap X_2 = \{0\} \text{ and } X = e_1X_1 + e_2X_2, \qquad (0.4)$$

where X_1 and X_2 are real linear spaces as well as $\mathbb{D}$-modules. Formula (0.4) is called the idempotent decomposition of X. Thus, any $x \in X$ can be uniquely written as $x = e_1x_1 + e_2x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. Further, if U and W be any two real linear spaces, then it can be shown that $X = e_1U + e_2W$ is a $\mathbb{D}$-module. Moreover, for any $\mathbb{D}$-module X, we denote the set of all zero-divisors in X by $\mathcal{NC}_X$, that is, $\mathcal{NC}_X = \{0 \neq z \in X : z \in e_1X \cup e_2X\}$.

Definition 2.1 *Let X be a $\mathbb{D}$-module and $\| \cdot \|_{\mathbb{D}} : X \to \mathbb{D}^+$ be a function such that for any $x, y \in X$ and $\alpha \in \mathbb{D}$, it satisfies the following properties:*

1. $\| x \|_{\mathbb{D}} = 0 \Leftrightarrow x = 0$.
2. $\| \alpha x \|_{\mathbb{D}} = |\alpha|_k \| x \|_{\mathbb{D}}$.
3. $\| x + y \|_{\mathbb{D}} \leq' \| x \|_{\mathbb{D}} + \| y \|_{\mathbb{D}}$.

Then we say that $\| \cdot \|_{\mathbb{D}}$ is a hyperbolic or $\mathbb{D}$-valued norm on X. The hyperbolic-valued norm on hyperbolic modules has been intensively discussed in [1, 12] and many other references therein.

Definition 2.2 *Let X be a $\mathbb{D}$-module of dimension greater 1. A map*

$$\langle \cdot,\cdot|_{\mathbb{D}} \rangle : X \times X \to \mathbb{D}$$

is said to be $\mathbb{D}$-valued 2-norm on X if for all $x, y, z \in X$ and $\alpha \in \mathbb{D}$ it satisfies the following properties:

1. $\| x, y \|_{\mathbb{D}} = 0$ if and only if x, y are linearly dependent,
2. $\| x, y \|_{\mathbb{D}} = \| y, x \|_{\mathbb{D}}$,

3. $\| \alpha x, y \|_{\mathbb{D}} = |\alpha|_k \| x, y \|_{\mathbb{D}}$,
4. $\| x+y, z \|_{\mathbb{D}} \leq' \| x, z \|_{\mathbb{D}} + \| y, z \|_{\mathbb{D}}$.

Then the pair $(X, \| .,. \|_{\mathbb{D}})$ is called a 2-normed $\mathbb{D}$-module. Further, it can be shown that $\| x, y \|_{\mathbb{D}} \in \mathbb{D}^+$ and $\| x, y+\alpha x \|_{\mathbb{D}} = \| x, y \|_{\mathbb{D}}$ $\forall\, x, y \in X$ and $\forall\, \alpha \in \mathbb{D}$.

2-inner product $\mathbb{D}$-modules

In this section, we introduce the notion of $\mathbb{D}$-valued 2-inner product on $\mathbb{D}$-modules and discuss some of its basic properties. We also discuss Parallelogram law and Polarization identity for 2-inner product $\mathbb{D}$-modules.

Definition 3.1 *Let X be a $\mathbb{D}$-module of dimension greater than 1. A map*

$\langle .,. | . \rangle : X \times X \times X \to \mathbb{D}$

is said to be a $\mathbb{D}$-valued 2-inner product on X if for each $x, y, z \in X$, it satisfies the following properties:

1. $\langle x, x | z \rangle \in \mathbb{D}^+$; $\langle x, x | z \rangle = 0$ if and only if x and z are linearly dependent,
2. $\langle x, x | z \rangle = \langle z, z | x \rangle$,
3. $\langle x, y | z \rangle = \langle y, x | z \rangle^*$
4. $\langle \alpha x, y | z \rangle = \alpha \langle x, y | z \rangle$; for any $\alpha \in \mathbb{D}$,
5. $\langle x + x', y | z \rangle = \langle x, y | z \rangle + \langle x', y | z \rangle$.

Then the pair $(X, \langle .,. | . \rangle)$ is called a 2-inner product $\mathbb{D}$-module. Further, for each $x, y, z \in X$ and for every $\alpha \in \mathbb{D}$, some basic properties of $\mathbb{D}$-valued 2-inner product $\langle .,. | . \rangle$ can be easily obtained as follows:

$$\langle x, \alpha y | z \rangle = \alpha^* \langle x, y | z \rangle \text{ and } \langle x, y | \alpha z \rangle = |\alpha|_k^2 \langle x, y | z \rangle.$$

Remark 3.2 *Let X_1 and X_2 be two real linear spaces with $\dim(X_1) > 1$ and $\dim(X_2) > 1$. In addition, we assume that both X_1 and X_2 are real 2-inner product spaces with corresponding 2-inner products $\langle .,. | . \rangle_1$ and $\langle .,. | . \rangle_2$. Let $X = e_1 X_1 + e_2 X_2$. Clearly, X is a $\mathbb{D}$-module with $\dim(X) > 1$*

.

For any $x = e_1x_1 + e_2x_2$, $y = e_1y_1 + e_2y_2$, $z = e_1z_1 + e_2z_2 \in X$, we define

$$\langle x, y \mid z\rangle = \langle e_1x_1 + e_2x_2, e_1y_1 + e_2y_2 \mid e_1z_1 + e_2z_2\rangle = e_1\langle x_1, y_1 \mid z_1\rangle_1 + e_2\langle x_2, y_2 \mid z_2\rangle_2. \quad (0.5)$$

Then the formula (0.5) is a $\mathbb{D}$-valued 2-inner product on X can be verified easily as follows :

Since $\langle x_l, x_l \mid z_l\rangle_l \geq 0$, $\forall \; x_l, z_l \in \mathbb{R}$, ($l = 1,2$), implies $\langle x, x \mid z\rangle \in \mathbb{D}^+$. Further,

$\langle x, x \mid z\rangle = 0 \Leftrightarrow e_1\langle x_1, x_1 \mid z_1\rangle_1 + e_2\langle x_2, x_2 \mid z_2\rangle_2 = 0 \Leftrightarrow \langle x_1, x_1 \mid z_1\rangle_1 = 0$ and $\langle x_2, x_2 \mid z_2\rangle_2 = 0$

$\Leftrightarrow x_1$ and z_1 arelinearl y dependenta nd x_2 and z_2 arelinearl y dependent

$\Leftrightarrow x$ and z are linearly dependent. Also, $\langle x, y \mid z\rangle = e_1\langle x_1, y_1 \mid z_1\rangle_1 + e_2\langle x_2, y_2 \mid z_2\rangle_2$

$$= e_1\langle y_1, x_1 \mid z_1\rangle_1 + e_2\langle y_2, x_2 \mid z_2\rangle_2 = e_1^*\langle y_1, x_1 \mid z_1\rangle_1^* + e_2^*\langle y_2, x_2 \mid z_2\rangle_2^*$$

$$= (e_1\langle y_1, x_1 \mid z_1\rangle_1 + e_2\langle y_2, x_2 \mid z_2\rangle_2)^* = \langle y, x \mid z\rangle^*.$$

Similarly, we can show $\langle x, x \mid z\rangle = \langle z, z \mid x\rangle$. Next, for any $\alpha \in \mathbb{D}$,

$$\langle \alpha\, x, y \mid z\rangle = \langle (e_1\alpha_1 + e_2\alpha_2)(e_1x_1 + e_2x_2), e_1y_1 + e_2y_2 \mid e_1z_1 + e_2z_2\rangle$$

$$= \langle e_1(\alpha_1\, x_1) + e_2(\alpha_2\, x_2), e_1y_1 + e_2y_2 \mid e_1z_1 + e_2z_2\rangle = e_1\langle \alpha_1\, x_1, y_1 \mid z_1\rangle_1 + e_2\langle \alpha_2\, x_2, y_2 \mid z_2\rangle_2$$

$$= e_1\alpha_1\langle x_1, y_1 \mid z_1\rangle_1 + e_2\alpha_2\langle x_2, y_2 \mid z_2\rangle_2 = (e_1\alpha_1 + e_2\alpha_2)\,(e_1\langle x_1, y_1 \mid z_1\rangle_1 + e_2\langle x_2, y_2 \mid z_2\rangle_2) = \alpha\langle x, y \mid z\rangle.$$

Finally, let $x, x', y, z \in X$. Then $\langle x + x', y \mid z\rangle = e_1\langle x_1 + x_1', y_1 \mid z_1\rangle + e_2\langle x_2 + x_2', y_2 \mid z_2\rangle$

$$= e_1(\langle x_1, y_1 \mid z_1\rangle + \langle x_1', y_1 \mid z_1\rangle) + e_2(\langle x_2, y_2 \mid z_2\rangle + \langle x_2', y_2 \mid z_2\rangle)$$

$$= (e_1\langle x_1, y_1 \mid z_1\rangle + e_2\langle x_2, y_2 \mid z_2\rangle) + (e_1\langle x_1', y_1 \mid z_1\rangle + e_2\langle x_2', y_2 \mid z_2\rangle)$$

$$= \langle x, y \mid z\rangle + \langle x', y \mid z\rangle.$$

Proposition 3.3 *Let X be a 2-inner product $\mathbb{D}$-module with $\dim(X) > 1$. Then $X_1 = e_1X$ and $X_2 = e_2X$ can be seen as 2-inner product real linear spaces with their real 2-inner products induced by the $\mathbb{D}$-valued 2-inner product on X.*

Proof. Let X be a 2-inner product $\mathbb{D}$-module with $\dim(X) > 1$. Clearly, e_1X and e_2X are real linear spaces with $\dim(e_1X) > 1$ and $\dim(e_2X) > 1$. Let $\langle .,. \mid .\rangle : X \times X \times X \to \mathbb{D}$ be the $\mathbb{D}$-valued 2-inner product on X. Then for any $x, y, z \in X$, we can write it as

$\langle x, y \mid z\rangle = e_1\Phi(x, y \mid z) + e_2\Psi(x, y \mid z),$

where $\Phi, \Psi : X \times X \times X \to \mathbb{R}$ are real-valued functions such that
$e_1\Phi(x, y \mid z) = e_1\langle x, y \mid z\rangle$ and $e_2\Psi(x, y \mid z) = e_2\langle x, y \mid z\rangle$. Further,

$$e_1\Phi(e_1x, e_1y \mid e_1z) + e_2\Psi(e_1x, e_1y \mid e_2z) = \langle e_1x, e_1y \mid e_1z\rangle = e_1\langle e_1x, e_1y \mid e_1z\rangle$$
$$= e_1\big(e_1\Phi(e_1x, e_1y \mid e_1z) + e_2\Psi(e_1x, e_1y \mid e_1z)\big) = e_1\Phi(e_1x, e_1y \mid e_1z).$$

This implies that

$$\Psi(e_1x, e_1y \mid e_1z) = 0 \text{ and } \langle e_1x, e_1y \mid e_1z\rangle = e_1\Phi(e_1x, e_1y \mid e_1z). \tag{0.6}$$

Similarly, one can show

$$\Phi(e_2x, e_2y \mid e_2z) = 0 \text{ and } \langle e_2x, e_2y \mid e_2z\rangle = e_2\Psi(e_2x, e_2y \mid e_2z). \tag{0.7}$$

Thus, by using (0.6) and (0.7), one can write

$$\langle x, y \mid z\rangle = e_1\langle x, y \mid z\rangle + e_2\langle x, y \mid z\rangle = e_1e_1^*e_1^2\langle x, y \mid z\rangle + e_2e_2^*e_2^2\langle x, y \mid z\rangle$$
$$= \langle e_1x, e_1y \mid e_1z\rangle + \langle e_2x, e_2y \mid e_2z\rangle = \langle e_1x, e_1y \mid e_1z\rangle + \langle e_2x, e_2y \mid e_2z\rangle$$
$$= e_1\Phi(e_1x, e_1y \mid e_1z) + e_2\Psi(e_2x, e_2y \mid e_2z).$$

That is, $\langle x, y \mid z\rangle = e_1\Phi(e_1x, e_1y \mid e_1z) + e_2\Psi(e_2x, e_2y \mid e_2z).$ (0.8)

We now show that Φ is a real 2-inner product on a real linear space e_1X and Ψ is a real 2-inner product on a real linear space e_2X. Since, for any $\lambda \in \mathbb{R}$ and $x, y, z \in X$, we have

$$\langle \lambda x, y \mid z\rangle = \lambda\langle x, y \mid z\rangle,$$

which gives that

$$e_1\Phi(\lambda e_1x, e_1y \mid e_1z) + e_2\Psi(\lambda e_2x, e_2y \mid e_2z) = \lambda(e_1\Phi(e_1x, e_1y \mid e_1z) + e_2\Psi(e_2x, e_2y \mid e_2z)).$$

Thus, $\Phi(\lambda e_1x, e_1y \mid e_1z) = \lambda\Phi(e_1x, e_1y \mid e_1z)$ and
$\Psi(\lambda e_2x, e_2y \mid e_2z) = \lambda\Psi(e_2x, e_2y \mid e_2z).$

Further, let $x, x', y, z \in X$. Then $\langle x + x', y \mid z\rangle = \langle x, y \mid z\rangle + \langle x', y \mid z\rangle$. Thus, by using (0.8), we obtain

$$e_1\Phi(e_1x + e_1x', e_1y \mid e_1z) + e_2\Psi(e_2x + e_2x', e_2y \mid e_2z) = e_1\Phi(e_1x, e_1y \mid e_1z)$$
$$+ e_2\Psi(e_2x, e_2y \mid e_2z) + e_1\Phi(e_1x', e_1y \mid e_1z) + e_2\Psi(e_2x', e_2y \mid e_2z)$$

which implies that

$\Phi(e_1x + e_1x', e_1y \mid e_1z) = \Phi(e_1x, e_1y \mid e_1z) + \Phi(e_1x', e_1y \mid e_1z)$ and
$\Psi(e_2x + e_2x', e_2y \mid e_2z) = \Psi(e_2x, e_2y \mid e_2z) + \Psi(e_2x', e_2y \mid e_2z).$

Next, for each $x,z \in X$, $\langle x,x\,|\,z\rangle = \langle z,z\,|\,x\rangle$. Thus, from (0.8), we get

$\Phi(e_1x,e_1x\,|\,e_1z)=\Phi(e_1z,e_1z\,|\,e_1x), \Psi(e_2x,e_2x\,|\,e_2z)=\Psi(e_2z,e_2z\,|\,e_2x).$ Similarly, we can show

$\Phi(e_1x,e_1y\,|\,e_1z)=\Phi(e_1y,e_1x\,|\,e_1z), \Psi(e_2x,e_2y\,|\,e_2z)=\Psi(e_2y,e_2x\,|\,e_2z).$

Now for any $x,z \in X$, we have $\langle x,x\,|\,z\rangle \in \mathbb{D}^+$.That is, $e_1\Phi(e_1x,e_1x\,|\,e_1z_1)+e_2\Psi(e_2x,e_2x\,|\,e_z)\in \mathbb{D}^+$. This implies that $\Phi(e_1x,e_1x\,|\,e_1z_1)\geq 0$ and $\Psi(e_2x,e_2x\,|\,e_z)\geq 0$. Finally, it remains to show that for any $x,z\in X$, $\Phi(e_1x,e_1x\,|\,e_1z)=0$ if and only if e_1x and e_1z are linearly dependent and similarly for Ψ. First suppose that $\Phi(e_1x,e_1x\,|\,e_1z)=0$. This means $e_1\Phi(e_1x,e_1x\,|\,e_1z)=0$ and hence by (0.6), we have $\langle e_1x,e_1x\,|\,e_1z\rangle=0$. Since $\langle .,.\,|.\rangle$ is a $\mathbb{D}$-valued 2-inner product on X implies that e_1x and e_1z are linearly dependent.Conversly, suppose that $x,z\in X$ such that e_1x and e_1z are linearly dependent. Then

$$\langle x,x\,|\,z\rangle = e_1\langle x,x\,|\,z\rangle + e_2\langle x,x\,|\,z\rangle = \langle e_1x,e_1x\,|\,e_1z\rangle + \langle e_2x,e_2x\,|\,e_2z\rangle$$
$$= \langle e_2x,e_2x\,|\,e_2z\rangle = e_2\Psi(e_2x,e_2x\,|\,e_2z).$$

Thus, by using (0.8), we have $\Phi(e_1x,e_1x\,|\,e_1z)=0$ and similarly for Ψ. Hence Φ defines a real 2-inner product on the real linear space e_1X and Ψ defines a real 2-inner product on the real linear space e_2X. On a 2-inner product $\mathbb{D}$-module $(X,\langle .,.\,|.\rangle)$, one may observe that $\|\ x,y\ \|_{\mathbb{D}} = \langle x,x\,|\,y\rangle^{\frac{1}{2}}$ defines a $\mathbb{D}$-valued 2-norm on X. Then it is easy to prove the following results for 2-inner product $\mathbb{D}$-module X.

Theorem 3.4 *Let $(X,\langle .,.\,|.\rangle)$ be a 2-inner product $\mathbb{D}$-module with dim$(X)>1$, where the $\mathbb{D}$-valued 2-inner product on X is induced by the inner products on its idempotent components X_1 and X_2. Then for any $x,y,z\in X$,*

$$\|\ x+y,z\ \|_{\mathbb{D}}^2 + \|\ x-y,z\ \|_{\mathbb{D}}^2 = 2(\|\ x,z\ \|_{\mathbb{D}}^2 + \|\ y,z\ \|_{\mathbb{D}}^2)$$

Proof. For any $x,y,z\in X$,

$$\|\ x+y,z\ \|_{\mathbb{D}}^2 = \langle x+y,x+y\,|\,z\rangle = \langle x_1e_1+x_2e_2+y_1e_1+y_2e_2, x_1e_1+x_2e_2+y_1e_1+y_2e_2\,|\,z_1e_1+z_2e_2\rangle$$
$$= \langle (x_1+y_1)e_1+(x_2+y_2)e_2, (x_1+y_1)e_1+(x_2+y_2)e_2\,|\,z_1e_1+z_2e_2\rangle$$
$$= e_1\langle (x_1+y_1),(x_1+y_1)\,|\,z_1\rangle_1 + e_2\langle (x_2+y_2),(x_2+y_2)\,|\,z_2\rangle_2$$

$= e_1\| x_1, z_1 \|_1^2 + e_1\| y_1, z_1 \|_1^2 + e_1\langle x_1, y_1 | z_1\rangle_1 + e_1\langle y_1, x_1 | z_1\rangle_1$

$+ e_2\| x_2, z_2 \|_2^2 + e_2\| y_2, z_2 \|_2^2 + e_2\langle x_2, y_2 | z_2\rangle_2 + e_2\langle y_2, x_2 | z_2\rangle_2.$

Similarly, we have

$\| x - y, z \|_{\mathbb{D}}^2 = \langle x - y, x - y | z\rangle$

$= e_1\| x_1, z_1 \|_1^2 - e_1\langle x_1, y_1 | z_1\rangle_1 - e_1\langle y_1, x_1 | z_1\rangle + e_1\| y_1, z_1 \|_1^2$

$+ e_2\| x_2, z_2 \|_2^2 - e_2\langle x_2, y_2 | z_2\rangle_2 - e_2\langle y_2, x_2 | z_2\rangle_2 + e_2\| y_2, z_2 \|_2^2.$

On adding we get,

$\| x + y, z \|_{\mathbb{D}}^2 + \| x - y, z \|_{\mathbb{D}}^2 = 2e_1\| x_1, z_1 \|_1^2 + 2e_1\| y_1, z_1 \|_1^2 + 2e_2\| x_2, z_2 \|_2^2 + 2e_2\| y_2, z_2 \|_2^2$

$= 2(e_1\| x_1, z_1 \|_1^2 + e_2\| x_2, z_2 \|_2^2) + 2(e_1\| y_1, z_1 \|_1^2 + e_2\| y_2, z_2 \|_2^2) = 2(\| x, z \|_{\mathbb{D}}^2 + \| y, z \|_{\mathbb{D}}^2).$

This proves the Parallelogram Law for 2-inner product $\mathbb{D}$-module X.

The next result is the Polarization Identity for 2-inner product $\mathbb{D}$-module X.

Theorem 3.5 *Let $(X, \langle .,. | . \rangle)$ be a 2-inner product $\mathbb{D}$-module with dim$(X) > 1$, where the $\mathbb{D}$-valued 2-inner product on X is induced by the inner products on its idempotent components X_1 and X_2. Then for any $x, y, z \in X$,*

$\langle x, y | z\rangle = \frac{1}{4}(\| x + y, z \|_{\mathbb{D}}^2 - \| x - y, z \|_{\mathbb{D}}^2)$. *Proof.* For any $x, y, z \in X$,

$\| x + y, z \|_{\mathbb{D}}^2 = \langle x + y, x + y | z\rangle = e_1\| x_1, z_1 \|_1^2 + e_1\| y_1, z_1 \|_1^2 + e_1\langle x_1, y_1 | z_1\rangle_1 + e_1\langle y_1, x_1 | z_1\rangle_1$

$+ e_2\| x_2, z_2 \|_2^2 + e_2\| y_2, z_2 \|_2^2 + e_2\langle x_2, y_2 | z_2\rangle_2 + e_2\langle y_2, x_2 | z_2\rangle_2.$

Similarly, we have

$\| x - y, z \|_{\mathbb{D}}^2 = \langle x - y, x - y | z\rangle = e_1\| x_1, z_1 \|_1^2 - e_1\langle x_1, y_1 | z_1\rangle_1 - e_1\langle y_1, x_1 | z_1\rangle_1 + e_1\| y_1, z_1 \|_1^2$

$+ e_2\| x_2, z_2 \|_2^2 - e_2\langle x_2, y_2 | z_2\rangle_2 - e_2\langle y_2, x_2 | z_2\rangle_2 + e_2\| y_2, z_2 \|_2^2.$

On subtracting we get,

$\| x + y, z \|_{\mathbb{D}}^2 + \| x - y, z \|_{\mathbb{D}}^2$

$$= e_1(2\langle x_1, y_1 | z_1\rangle_1 + 2\langle y_1, x_1 | z_1\rangle_1) + e_2(2\langle x_2, y_2 | z_2\rangle_2 + 2\langle y_2, x_2 | z_2\rangle_2)$$

$$= 2(e_1\langle x_1, y_1 | z_1\rangle_1 + e_2\langle x_2, y_2 | z_2\rangle_2) + 2(e_1\langle y_1, x_1 | z_1\rangle_1 + e_2\langle y_2, x_2 | z_2\rangle_2)$$

$$= 2\langle x, y \mid z\rangle + 2\langle y, x \mid z\rangle = 2\langle x, y \mid z\rangle + 2\langle x, y \mid z\rangle^*$$

$$= 4\langle x, y \mid z\rangle, \text{ because } \langle x, y \mid z\rangle \in \mathbb{D}, \text{ so } \langle x, y \mid z\rangle^* = \langle x, y \mid z\rangle.$$

Hence $\langle x, y \mid z\rangle = \frac{1}{4}(\| x+y, z \|_{\mathbb{D}}^2 - \| x-y, z \|_{\mathbb{D}}^2)$.

Now, it is natural to study the relation between the $\mathbb{D}$-valued 2-inner product and $\mathbb{D}$-valued inner product on $\mathbb{D}$-modules. In this direction, C. Diminnie, S. Gahler and A. White ([4], [5]) are probably the first to draw a connection between real inner product and real 2-inner product on a real linear space. They proved that if X is a real inner product space, then the real 2-inner product can be defined on X. Further, H. Gunawan [8] proved that a real 2-inner product space is a real inner product space. However, with the little adjustments, a similar relation between $\mathbb{D}$-valued 2-inner product and $\mathbb{D}$-valued inner product on $\mathbb{D}$-modules can be developed.

For this, let $\langle .,. \rangle$ be a $\mathbb{D}$-valued inner product on a $\mathbb{D}$-module X. Now for any $x, y, z \in X$, we define

$$\langle x, y \mid z\rangle = \langle x, y\rangle\langle z, z\rangle - \langle x, z\rangle\langle y, z\rangle. \tag{0.9}$$

Then the formula (0.9) is a $\mathbb{D}$-valued 2-inner product on a $\mathbb{D}$-module X can be verified easily as follows:

$$\langle x, x \mid z\rangle = \langle x, x\rangle\langle z, z\rangle - \langle x, z\rangle\langle x, z\rangle = \langle x, x\rangle\langle z, z\rangle - \langle x, z\rangle\langle x, z\rangle^* = \| x \|_{\mathbb{D}}^2 \| z \|_{\mathbb{D}}^2 - |\langle x, z\rangle|_k^2 \in \mathbb{D}^+,$$

(byusingthe bicomplexS chwarzineq uality).

Also, $\langle x, x \mid z\rangle = 0 \Leftrightarrow \langle x, x\rangle\langle z, z\rangle - \langle x, z\rangle\langle x, z\rangle = 0 \Leftrightarrow \| x \|_{\mathbb{D}}^2 \| z \|_{\mathbb{D}}^2 - |\langle x, z\rangle|_k^2 = 0$

$\Leftrightarrow$ either $x = 0$ or $z = 0 \Leftrightarrow x$ and z arelinearl ydependent .

Next, $\langle x, y \mid z\rangle = \langle x, y\rangle\langle z, z\rangle - \langle x, z\rangle\langle y, z\rangle = \langle y, x\rangle^*\langle z, z\rangle - \langle y, z\rangle\langle x, z\rangle = \langle y, x\rangle^*\langle z, z\rangle^* - \langle y, z\rangle^*\langle x, z\rangle^*$

(because $\langle z, z\rangle, \langle y, z\rangle$ and $\langle x, z\rangle$ arehyperbo licnumbers)

$= (\langle y, x\rangle\langle z, z\rangle - \langle y, z\rangle\langle x, z\rangle)^*$

$= \langle y, x \mid z\rangle^*$. Similarly, we can prove $\langle x, x \mid z\rangle = \langle z, z \mid x\rangle$. Further, for any $\alpha \in \mathbb{D}$,

$$\langle \alpha x, y \mid z\rangle = \langle \alpha x, y\rangle\langle z, z\rangle - \langle \alpha x, z\rangle\langle y, z\rangle$$

$$= \alpha\langle x, y\rangle\langle z, z\rangle - \alpha\langle x, z\rangle\langle y, z\rangle$$

$$=\alpha(\langle x,y\rangle\langle z,z\rangle-\langle x,z\rangle\langle y,z\rangle)=\alpha\langle x,y\,|\,z\rangle.$$

Finally, for each $x, x', y, z \in X$, we have

$$\langle x+x',y\,|\,z\rangle=\langle x+x',y\rangle\langle z,z\rangle-\langle x+x',z\rangle\langle y,z\rangle=(\langle x,y\rangle+\langle x',y\rangle)\langle z,z\rangle-(\langle x,z\rangle+\langle x',z\rangle)\langle y,z\rangle$$

$$=\langle x,y\rangle\langle z,z\rangle+\langle x',y\rangle\langle z,z\rangle-\langle x,z\rangle\langle y,z\rangle-\langle x',z\rangle\langle y,z\rangle$$

$$=(\langle x,y\rangle\langle z,z\rangle-\langle x,z\rangle\langle y,z\rangle)+(\langle x',y\rangle\langle z,z\rangle-\langle x',z\rangle\langle y,z\rangle)=\langle x,y\,|\,z\rangle+\langle x',y\,|\,z\rangle.$$

Next, we will show that every $\mathbb{D}$-valued 2-inner product module is a $\mathbb{D}$-valued inner product module. For this, let $(X,\langle .,.|.\rangle)$ be a $\mathbb{D}$-valued 2-inner product module. Choose a,b in X so that a and b are linearly independent. In addition, we assume that a_l is linearly independent to b_l, $l=1,2$. Then we define a function $\langle .,.\rangle : X\times X\to\mathbb{D}$ as follows:

$$\langle x,y\rangle=\langle x,y\,|\,a\rangle+\langle x,y\,|\,b\rangle,\ \forall\, x,y\in X. \qquad (0.10)$$

The formula (0.10) is a $\mathbb{D}$-valued inner product on a $\mathbb{D}$-module X. To see this, let $x,y\in X$. Then, clearly $\langle x,x\rangle=\langle x,x\,|\,a\rangle+\langle x,x\,|\,b\rangle\in\mathbb{D}^+$. Further,

$$\langle x,x\rangle=0\Leftrightarrow\langle x,x\,|\,a\rangle+\langle x,x\,|\,b\rangle=0$$

$\Leftrightarrow x,a$ arelinearl ydependent and x,b
are linearlyde pendent

$\Leftrightarrow x=0,$ becauseaan dbarelinea rlyindepen dent.

In order to proceed to the next step, take $\alpha\in\mathbb{D}$ and $x,y\in X$. Then

$$\langle\alpha x,y\rangle=\langle\alpha x,y\,|\,a\rangle+\langle\alpha x,y\,|\,b\rangle=\alpha\langle x,y\,|\,a\rangle+\alpha\langle x,y\,|\,b\rangle=\alpha(\langle x,y\,|\,a\rangle+\langle x,y\,|\,b\rangle)=\alpha\langle x,y\rangle.$$

Also, $\langle x,y\rangle=\langle x,y\,|\,a\rangle+\langle x,y\,|\,b\rangle=\langle y,x\,|\,a\rangle^*+\langle y,x\,|\,b\rangle^*=\langle y,x\rangle^*.$

Finally, for any $x,x',y,z\in X$, we have $\langle x+x',y\rangle=\langle x+x',y\,|\,a\rangle+\langle x+x',y\,|\,b\rangle$

$$=\langle x,y\,|\,a\rangle+\langle x',y\,|\,a\rangle+\langle x,y\,|\,b\rangle+\langle x',y\,|\,b\rangle$$

$$=(\langle x,y\,|\,a\rangle+\langle x,y\,|\,b\rangle)+(\langle x',y\,|\,a\rangle+\langle x',y\,|\,b\rangle)$$

$$=\langle x,y\rangle+\langle x',y\rangle.$$

References

[1] D. Alpay, M. E. Luna-Elizarraras, M. Shapiro and D. C. Struppa, *Basics of Functional Analysis with Bicomplex scalars, and Bicomplex Schur Analysis*, Springer Briefs in Mathematics, 2014.

[2] Y. J. Cho, K. S. Ha and W. S. Kim, *Strictly convex linear 2-normed spaces*, Math. Japonica, **26**, (1981), 475-478.

[3] C. Diminnie, S. Gahler and A. White, *Strictly convex linear 2-normed spaces*, Math. Nachr. **59**, (1974), 319-324.

[4] C. Diminnie, S. Gahler and A. White, *Remarks on generalization of 2-inner products*, Math. Nachr. **74**, (1976), 363-372.

[5] C. Diminnie, S. Gahler and A. White, *2-inner product spaces II*, Demonstration Math. **10** (1977), 169-188.

[6] R. Ehret, Linear 2-normed spaces, Dissertation, St. Louis University, 1969.

[7] S. Gahler, *Lineare 2-normierte Raume*, Math. Nachr. **28**, (1965), 1-43.

[8] H. Gunawan, *Inner products on n-inner product spaces*, Soochow Journal of Math. **28** (2002), No. 4, 389-398.

[9] S. N. Lal, S. Bhattacharya and C. Sreedhar, *Complex 2-normed linear spaces and extension of linear 2-functionals*, J. Anal. Appl. **20**, (2001), 35-53.

[10] Z. Lewandowska, *Linear operators on generalized 2-normed spaces*, Bull. Math. Soc. Sci. Math. Roumanie, **42**, (1999), 353-368.

[11] Z. Lewandowska, *Bounded 2-linear operators on 2-normed sets*, Glasnik Mat., **39**, (2004), 303-314.

[12] M. E. Luna-Elizarraras, C. O. Perez-Regalado and M. Shapiro, *On linear functionals and Hahn-Banach theorems for hyperbolic and bicomplex modules*, Adv. Appl. Clifford Algebr. **24**, (2014), 1105-1129.

[13] D. Rochon and M. Shapiro, *On algebraic properties of bicomplex and hyperbolic numbers*, Anal. Univ. Oradea, Fasc. Math. **11** (2004), 71-110.

[14] K. Singh and R. Kumar, *2-normed* $\mathbb{D}$ *-modules and Hahn-Banach theorem for* $\mathbb{D}$ *-linear 2-functionals*, Anal. Univ. Oradea, Fasc. Math. **23** (2016), 117-126.

[15] A. White, *2-Banach spaces*, Math. Nachr. **42**, (1969), 43-60.

[16] A. White, Y. J. Cho and S. S. Kim, *Characterizations of 2-inner product spaces*, Proceedings of Nonlinear Functional Analysis and Applications, **2** (1997), 145-152.