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Composite Convolution Operators on $L^2(\mathbb{R})$

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Abstract: This paper is the study of composite convolution operators on $L^2(\mathbb{R})$. Bounded and Hermitian composite convolution operators are characterized. Adjoint of the composite convolution operator is computed.

Key words: *Convolution product, Hermitian operator, Compact operator, Hilbert-Schmidt operator.*
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Introduction

If $\phi \in L^1(\mathbb{R})$, $f \in L^2(\mathbb{R})$, then we form the convolution product $f \otimes \phi$ which is defined by $(f \otimes \phi)(x) = \int_{y=-\infty}^{\infty} f(y)\phi(x-y)d\mu(y)$. If $T:\mathbb{R} \to \mathbb{R}$ is a mapping such that the transformation $C_{T,\phi}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by $(C_{T,\phi}f)(x) = (f \otimes \phi)((T(x)))$ is bounded. We shall call $C_{T,\phi}$ a composite convolution operator induced by the pair (ϕ, T). For literature concerning composite operators and convolution operators, we refer to Singh and Komal [8,9], Komal and Gupta [6], Carlson [1], Stepanov[10], Singh, Gupta and Komal [7].

Bounded Composite Convolution Operators

In this section we characterize bounded composite convolution operators.

Theorem 2.1 *Let $T:\mathbb{R} \to \mathbb{R}$ be a non-singular measurable mapping and $\phi \in L^1(\mathbb{R})$. Then $C_{T,\phi}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded operator if $\exists M > 0$ such that $f_0(x) \le M$ for μ-almost all $x \in \mathbb{R}$, where f_0 is the Radon-Nikodym derivative of the measure μT^{-1} w.r.t. the measure μ.*

Proof.

$$\| C_{T,\phi} f \|^2 = \int |(f \otimes \phi)(T(x))|^2 \, d\mu(x) = \int | f \otimes \phi(x)|^2 \, d\mu T^{-1}(x)$$

$$= \int \left| \int f(x-y)\phi(y)d\mu(y) \right|^2 f_0(x)d\mu(x) \le \int f_0(x)\left[\int |(\tau_x f)(y)||\phi(y)| \, d\mu(y)\right]^2 d\mu(x)$$

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$= \int f_0(x)\left[\int |(\tau_x f)(y)|\, d\lambda(y)\right]^2 d\mu(x)$, where $\lambda(E) = \int_E |\phi(y)|\, d\mu(y)$ for every measurable subset E of R,

$$= \int f_0(x)\left[\int |\chi_{R}(y)(\tau_x f)(y)|\, d\lambda(y)\right]^2 d\mu(x)$$

$$\leq \int f_0(x)\left[\int |\chi_{R}(y)|^2\, d\lambda(y)\int |(\tau_x f)(y)|^2\, d\lambda(y)\right] d\mu(x)$$

$$= \int f_0(x)\left(\int |\phi(y)|\, d\mu(y)\int |(\tau_x f)(y)|^2\, d\lambda(y)\right) d\mu(x) = \int f_0(x)\left(\|\phi\|_1 \int |(\tau_x f)(y)|^2 |\phi(y)|\, d\mu(y)\right) d\mu(x)$$

$$= \|\phi\|_1 \int |\phi(y)|\left[\int f_0(x)|(\tau_x f)(y)|^2\, d\mu(x)\right] d\mu(y) \leq M\|\phi\|_1 \int_{R} |\phi(y)|\left(\int_{R} |f(x-y)|^2\, d\mu(x)\right) d\mu(y)$$

$$= M\|\phi\|_1 \|f\|_2^2 \int |\phi(y)|\, d\mu(y) = M\|\phi\|_1 \|f\|_2^2 \|\phi\|_1 = M\|\phi\|_1^2 \|f\|_2^2 .$$

Hence,

$\|C_{T,\phi} f\|^2 \leq \sqrt{M}\, \|f\|_2 \|\phi\|_1$, for every $f \in L^2(R)$. This proves that $C_{T,\phi}$ is a bounded operator.

Example 2.2

Let $\phi : R \to R$ be defined by

$$\phi(x) = \begin{cases} 1 & \text{for } x < 1/2 \\ 0 & \text{for } x \geq 1/2 \end{cases}$$

Then $\int_{R} |\phi(x)|\, dx = \int_{-1/2}^{1/2} dx = 1$, so that $\phi \in L^1(R)$. Let $T : R \to R$ be defined by $T(x) = x + 1 \forall x \in R$.

Then $f_0(x) = \dfrac{d\mu T^{-1}(x)}{d\mu(x)} = 1.$

Therefore, $\|C_{T,\phi} f\|^2 = \int_{R} |(f \otimes \phi)T(x)|^2\, d\mu(x) = \int_{R} f_0(x)|(f \otimes \phi)(x)|^2\, d\mu(x) = \int_{R} |(f \otimes \phi)(x)|^2\, d\mu(x)$

$$= \int_{R} \left|\int_{R} f(x-y)\phi(y)\, d\mu(y)\right|^2 d\mu(x) = \int_{R} \left|\int \chi_{[-1/2,1/2]}(y) f(x-y)\, d\mu(y)\right|^2 d\mu(x)$$

$$\leq \int_{R}\left[\int \chi_{[-1/2,1/2]}(y)\, d\mu(y) \int_{-1/2}^{1/2} |f(x-y)|^2\, d\mu(y)\right] d\mu(x) = \int_{-\infty}^{\infty}\int_{-1/2}^{1/2} |f(x-y)|^2\, d\mu(y)\, d\mu(x)$$

$$= \int_{-\infty}^{\infty}\left[\int_{x-1/2}^{x+1/2} |f(t)|^2\, d\mu(t)\right] d\mu(x) = \int_{-\infty}^{\infty}\left[\int_{t-1/2}^{t+1/2} |f(t)|^2\, d\mu(x)\right] d\mu(t) = \int_{-\infty}^{\infty} |f(t)|^2 \left[\int_{t-1/2}^{t+1/2} d\mu(x)\right] d\mu(t)$$

$= \int_{-\infty}^{\infty} |f(t)|^2\, d\mu(t) = \|f\|^2$. Hence, $\|C_{T,\phi} f\| \leq \|f\| \quad \forall f \in L^2(R)$.

This proves that $C_{T,\phi}$ is a bounded operator. W

Hermitian Composite Convolution Operators

In this section, we compute the adjoint of the composite convolution operator on $L^2(\mathbb{R})$. We also characterize Hermitian composite convolution operators. An example is given for illustration.

For $\phi \in L^1(\mathbb{R})$, $g \in L^2(\mathbb{R})$, we define $A_\phi g = \phi^* \otimes (E(g)\circ T^{-1}) f_0$, where $\phi^*(x) = \overline{\phi(-x)}$.

In the following theorem, we prove that A_ϕ is the adjoint of composite convolution operator $C_{T,\phi}$.

Theorem 3.1 *Let* $C_{T,\phi} \in B(L^2(\mathbb{R}))$. *Then* $C^*_{T,\phi} = A_\phi$.

Proof. Let $f, g \in L^2(\mathbb{R})$. Consider $\langle C_{T,\phi} f, g\rangle = \int_{\mathbb{R}} (f \otimes \phi) T(x) \overline{g(x)} d\mu(x)$

$$= \int_{\mathbb{R}} (f\otimes\phi)(x) \overline{E(g)} \circ T^{-1}(x) f_0(x) d\mu(x) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(y) \phi(x-y) E(\bar{g}) \circ T^{-1}(x) f_0(x) d\mu(y) \right] d\mu(x)$$

$$= \int_{\mathbb{R}} f(y) \left[\int_{\mathbb{R}} \overline{\overline{\phi(x-y)} E(g\circ T^{-1})(x) f_0(x) d\mu(x)} \right] d\mu(y) = \int_{\mathbb{R}} f(y) \left[\int_{\mathbb{R}} \overline{\phi^*(y-x) E(g\circ T^{-1})(x) f_0(x) d\mu(x)} \right] d\mu(y)$$

$$= \int_{\mathbb{R}} f(y) \left(\overline{\phi^* \otimes (E(g)\circ T^{-1} f_0)(y)} \right) d\mu(y) = \int_{\mathbb{R}} f(y) \left(\overline{A_\phi g} \right)(y) d\mu(y) = \langle f, A_\phi g\rangle \ \forall f, g \in L^2(\mathbb{R}).$$

Hence,

$C^*_{T,\phi} = A_\phi$.

Theorem 3.2 *Let* $C_{T,\phi} \in B(L^2(\mathbb{R}))$. *Then* $C_{T,\phi}$ *is Hermitian if* $\phi = \phi^*$.

Proof. We first assume that the condition is true. Then for $f, g \in L^2(\mathbb{R})$,consider

$$\langle C^*_{T,\phi} f, g\rangle = \langle f, C_{T,\phi} g\rangle = \int f(x) \overline{(C_{T,\phi} g)}(x) d\mu(x)$$

$$= \int f(x) \overline{(g\otimes\phi)} T(x) d\mu(x) = \int f_0(x) E(f)\circ T^{-1}(x) \overline{(g\otimes\phi)(x)} d\mu(x)$$

$$= \int f_0(x) E(f)\circ T^{-1}(x) \left(\int \overline{\phi(x-y)}\, \overline{g(y)} d\mu(y) \right) d\mu(x) = \int \overline{g(y)} \left[\int \overline{\phi(x-y)} f_0(x) E(f) \circ T^{-1}(x) d\mu(x) \right] d\mu(y)$$

$$= \int \overline{g(y)} \left[\int \phi^*(y-x) f_0(x) E(f)\circ T^{-1}(x) d\mu(x) \right] d\mu(y) = \int \overline{g(y)} \left(\phi^* \otimes E(f)\circ T^{-1} f_0 \right)(y) d\mu(y)$$

$= \int \overline{g(y)} \left(\phi \otimes E(f)\circ T^{-1} f_0 \right)(y) d\mu(y)$ [because $\phi = \phi^*$] $= \langle C_{T,\phi} f, g\rangle$. This proves that $C_{T,\phi}$ is Hermitian.

Example 3.3

Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by $\phi(x) = e^{-x^2/2}$ forall $x \in \mathbb{R}$. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = 1 - x$ for μ – almostall $x \in \mathbb{R}$. Then $\phi \in L^1(\mathbb{R})$ and

$\int_{-\infty}^{\infty}|\phi(x)|\,dx=\int_{-\infty}^{\infty}e^{-x^2/2}dx=\sqrt{2\pi}$ and $f_0(y)=\frac{d\mu T^{-1}}{d\mu}(y)=1$. Now

$\phi(x-y)=e^{-\frac{1}{2}(x-y)^2}$ and $\phi(y-x)=e^{-\frac{1}{2}(y-x)^2}$ Therefore, $\phi(x-y)=\overline{\phi(y-x)}=\phi^*(x-y)$.

So, $(C^*_{T,\phi}f)(x)=(\phi^*\otimes E(f)\circ T^{-1}f_0)(x)=\int_{-\infty}^{\infty}\phi^*(x-y)\,E(f)\circ T^{-1}(y)f_0(y)d\mu(y)$

$=\int_{-\infty}^{\infty}\phi(y-x)\,(f\circ T)(y)f_0(y)d\mu(y)=\int_{-\infty}^{\infty}\phi(y-x)\,f(1-y)(1)d\mu(y)=\int_{-\infty}^{\infty}\phi(1-t-x)\,f(t)d\mu(t)$ (3.1) and

$(C_{T,\phi}f)(x)=(f\otimes\phi)(T(x))=\int_{-\infty}^{\infty}f(y)\phi(T(x)-y)d\mu(y)=\int_{-\infty}^{\infty}f(y)\phi(1-x-y)d\mu(y)$

$=\int_{-\infty}^{\infty}f(t)\phi(1-t-x)d\mu(t)$ (3.2)

From (3.1) and (3.2), we get $C_{T,\phi}f=C^*_{T,\phi}f$ for every $f\in L^2(\mathbb{R})$. Hence, $C_{T,\phi}$ is Hermitian. W

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