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Eneström - Kakeya Theorem and Zero-free Regions of Polynomials

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Abstract: In the literature, some extensions and generalizations of Eneström-Kakeya theorem are available. In this paper we refine some known results and accordingly obtain the zero –free regions of polynomials.

Keywords:-*Polynomials; Eneström-Kakeya Theorem; Zero.* ***Mathematics Subject Classification***:- *30A10, 30C10,30C15.*

Introduction and Statement of Results

In the theory of distribution of zeros of polynomials, the Eneström-Kakeya theorem [4] given below in theorem A is a well known result.

Theorem A . Let P(z) = $\sum_{i=0}^{n} a_i\, z^i$ be a polynomial of degree n such that

$a_n \geq a_{n-1} \geq \ldots\ldots \geq a_1 \geq a_0 > 0$ Then all the zeros of P(z) lie in the disk $|z| \leq 1$.

Many attempts have been made to extend and generalize the Eneström-Kakeyatheorem . A. Joyal et al [3] extended the Eneström-Kakeya theorem to the polynomials with general monotonic coefficients by proving that if $a_n \geq a_{n-1} \geq \ldots\ldots \geq a_1 \geq a_0$ Then P(z) has all its zeros in the disk

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}$$

Further Aziz and Zargar [1] generalized the result of A.Joyal et al [3] and the Eneström-Kakeya theorem as given below in theorem B.

Theorem B. Suppose P(z) = $\sum_{i=0}^{n} a_i\, z^i$ be a polynomial of degree n such that

For some $\lambda \geq 1$, $\lambda a_n \geq a_{n-1} \geq \ldots\ldots \geq a_1 \geq a_0$ then all the zeros of P(z) $= \sum_{i=0}^{n} a_i\, z^i$, lie in the disk

$$|z + (\lambda - 1)| \leq \frac{\lambda a_n - a_0 + |a_0|}{|a_n|}$$

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Theorem C. Suppose P(z) = $\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that either $a_n \geq a_{n-2} \geq \ldots \geq a_2 \geq a_0 > 0$ and $a_{n-1} \geq a_{n-3} \geq \ldots \geq a_3 \geq a_1 > 0$ if n is even Or $a_n \geq a_{n-2} \geq \ldots \geq a_3 \geq a_1 > 0$ and $a_{n-1} \geq a_{n-3} \geq \ldots \geq a_2 \geq a_0 > 0$ if n is odd , then all the zeros of P(z) lie in the disk $\left|z + \frac{a_{n-1}}{a_n}\right| \leq 1 + \frac{a_{n-1}}{a_n}$ But Govil and Rahman [2] proved that if , P(z) = $\sum_{i=0}^{n} a_i z^i$ is a complex polynomial of degree n with $|arg. a_i - \beta| \leq \alpha \leq \frac{\pi}{2}$, (i= 0,1,2,…….n) for some β real and $|a_n| \geq |a_{n-1}| \geq \ldots \geq |a_1| \geq |a_0|$, then P(z) has all its zeros in the disk $|z| \leq \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{i=0}^{n-1}|a_i|$Again Shah and Liman [5] generalized theorem B and the result of Govil and Rahman[2] and proved theorems D and Eas stated below.

Theorem D. Suppose P(z) = $\sum_{i=0}^{n} a_i z^i$ be a complex polynomial of degree n with Re(a_i)=α_i and Im(a_i)=β_i , i=0,1,2,……..,n. If for some $\lambda \geq 1$

$\lambda\alpha_n \geq \alpha_{n-1} \geq \ldots \geq \alpha_1 \geq \alpha_0$,$\beta_n \geq \beta_{n-1} \geq \ldots \geq \beta_1 \geq \beta_0 > 0$Then P (z) has all its zeros in the disk

$$\left|z + \frac{(\lambda - 1)\alpha_n}{a_n}\right| \leq \frac{\lambda\alpha_n - \alpha_0 + |\alpha_0| + \beta_n}{|a_n|}$$

Theorem E. Suppose P (z) = $\sum_{i=0}^{n} a_i z^i$ be a complex polynomial of degree n such that $|arg. a_i - \beta| \leq \alpha \leq \frac{\pi}{2}$, (i= 0,1,2,…….n) for some β real and for some $\lambda \geq 1$

$\lambda_1|a_n| \geq |a_{n-2}| \geq \ldots \geq |a_2| \geq |a_0|$ Then P(z) has all its zeros in the disk

$$|z + (\lambda - 1)| \leq \frac{1}{|a_n|}\{ (\lambda|a_n| - |a_0|)(\cos\alpha + \sin\alpha) + |a_0| + 2\sin\alpha \sum_{i=0}^{n-1}|a_i|\}$$

The main purpose of this paper is to refine some results mentioned above and define the zero -free regions of polynomials in theorems C, D and E.

Theorems and Proofs

Theorem 1 Suppose P(z) = $\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree n such that either $a_n \geq a_{n-2} \geq \ldots \geq a_2 \geq a_0$ and $a_{n-1} \geq a_{n-3} \geq \ldots \geq a_3 \geq a_1$ if n is even Or, $a_n \geq a_{n-2} \geq \ldots \geq a_3 \geq a_1$ and $a_{n-1} \geq a_{n-3} \geq \ldots \geq a_2 \geq a_0$ if n is oddThen P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_n| + a_n + |a_{n-1}| + a_{n-1} + |a_1| - a_1 - a_0}$$

Proof To prove the theorem , we consider a polynomial F(z) defined by

F(z) = (1 - z^2) P(z) = (1 - z^2) ($a_0 + a_1$z + a_2z^2 +……+$a_{n-1}z^{n-1}$+ a_nz^n) = - a_nz^{n+2} - $a_{n-1}z^{n+1}$ + (a_n - a_{n-2})z^n + (a_{n-1} - a_{n-3})z^{n-1} + …..+ (a_3 - a_1)z^3 + (a_2 - a_0)z^2 + $a_1z + a_0$ = g(z) + a_0, where g(z) = - a_nz^{n+2} - $a_{n-1}z^{n+1}$ + (a_n - a_{n-2})z^n + (a_{n-1} - a_{n-3})z^{n-1} + …..+ (a_3 - a_1)z^3 + (a_2 - a_0)z^2 + a_1z If $|z| < 1$ then $|g(z)| \leq |a_n| + |a_{n-1}|$ + (a_n - a_{n-2}) + (a_{n-1} - a_{n-3}) + …..+ (a_3 - a_1)+ (a_2 - a_0) + $|a_1|$ since by hypothesis$a_n \geq a_{n-2} \geq$ …..$\geq a_2 \geq a_0$ and $a_{n-1} \geq a_{n-3} \geq$ …..$\geq a_3 \geq a_1$ .On simplification, we have

$$|g(z)| \leq |a_n| + |a_{n-1}| + a_n + a_{n-1} - a_1 - a_0 + |a_1|$$

Also we have, g(0) = 0, therefore by Schwarz lemma , it follows that $|g(z)| \leq M|z|$for $|z| < 1$ where M = $|a_n| + |a_{n-1}| + a_n + a_{n-1} - a_1 - a_0 + |a_1|$

Again for $|z| < 1$, we have $|F(z)|$ =| g(z) + a_0| = $|a_0 + g(z)| \geq |a_0|$ - $|g(z)| \geq |a_0|$ - $M|z| > 0$ if$|a_0| > M|z|$I.e. if $|z| < \frac{|a_0|}{M}$ Also we can show that M$\geq |a_0|$ as $|z| < 1$ and hence the desired result follows.

Theorem 2 Let P (z) = $\sum_{i=0}^{n} a_i\ z^i$ be a complex polynomial of degree n with Re(a_i)=α_i and Im(a_i)=β_i , i=0,1,2,……..,n . Let for some $\lambda \geq 1$, $\lambda\alpha_n \geq \alpha_{n-1} \geq$ …..$\geq \alpha_1 \geq \alpha_0$, $\beta_n \geq \beta_{n-1} \geq$ …..$\geq \beta_1 \geq \beta_0$, then P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_n| + (\lambda - 1)|\alpha_n| + \lambda\alpha_n - \alpha_o + \beta_n - \beta_0}$$

Proof To prove the theorem, we consider a polynomial F(z) defined by

F(z) =(1 – z) P(z) = (1 – z) $\sum_{i=0}^{n} a_i\ z^i$ = (1 – z)($a_0 + a_1$z + a_2z^2 +……+$a_{n-1}z^{n-1} + a_nz^n$)

On simplification , we have F(z) = $-a_nz^{n+1}$ +(a_n - a_{n-1})z^n + ……….+(a_1 - a_0)z + a_0

= g(z) +a_0, where g(z) =$-a_nz^{n+1}$ +(a_n - a_{n-1})z^n + ……….+(a_1 - a_0)z

Using hypothesis, we can write g(z) as

g(z)=$-a_nz^{n+1}$ + ($\alpha_n - \alpha_{n-1}$)z^n + ($\alpha_{n-1} - \alpha_{n-2}$)z^{n-1} +………+($\alpha_1 - \alpha_0$)z +

i { $(\beta_n - \beta_{n-1})z^n$+ $(\beta_{n-1} - \beta_{n-2})z^{n-1}$ +........+ $(\beta_1 - - \beta_0)$z}
$= -a_n z^{n+1} - (\lambda - 1)\alpha_n z^n + (\lambda\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})\, z^{n-1} + \ldots\ldots$
+$(\alpha_1 - \alpha_0)$z + i { $(\beta_n - \beta_{n-1})z^n$ + $(\beta_{n-1} - \beta_{n-2})z^{n-1}$ +........+ $(\beta_1 - - \beta_0)$z}
Now if$|z| < 1$,then on simplification , we have

$$|g(z)| \le |a_n| + (\lambda - 1)|\alpha_n| + \lambda\alpha_n - \alpha_o + \beta_n - \beta_0$$

From above , g(0) = 0, therefore by Schwarz lemma ,it follows that
$|g(z)| \le M|z|$for$|z| < 1$, where M= $|a_n| + (\lambda - 1)|\alpha_n| + \lambda\alpha_n - \alpha_o + \beta_n - \beta_0$
Again for$|z| < 1$,$|F(z)|$ = |g(z) $+ a_0| \ge |a_0| - |g(z)| \ge |a_0| - M|z| > 0$, if$|a_0| > M|z|$
i.e., if$|z| < \frac{|a_0|}{M}$where M = $|a_n| + (\lambda - 1)|\alpha_n| + \lambda\alpha_n - \alpha_o + \beta_n - \beta_0$Also we can show that M $\ge |a_0|$ as $|z| < 1$Hence the desired result follows.

Theorem 3.. Suppose P(z) = $\sum_{i=0}^{n} a_i\, z^i$ be a complex polynomial of degree n such that $|arg.a_i - \beta| \le \alpha \le \frac{\pi}{2}$, (i= 0,1,2,.......n) for some β real and for some $\lambda \ge 1$
$\lambda|a_n| \ge |a_{n-1}| \ge \ldots\ldots\ldots \ge |a_1| \ge |a_0|$Then P(z) does not vanish in the disk
$|z| <$

$$\frac{|a_0|}{\lambda|a_n| + (\lambda|a_n| - |a_0|)\cos\alpha + (\lambda|a_n| + |a_0|)\sin\alpha + 2\sin\alpha\sum_{i=0}^{n-1}|a_i|}$$

Proof To prove the theorem , we consider a polynomial F(z) defined by
F(z) = (1 – z) P(z)= (1 – z) ($a_0 + a_1$z + $a_2 z^2$ +......+$a_{n-1}z^{n-1}$ + $a_n z^n$) = $-a_n z^{n+1}$+(a_n - a_{n-1})z^n + ...+(a_1 - a_0)z + a_0 $= -a_n z^{n+1} - (\lambda - 1)a_n z^n + (\lambda a_n - a_{n-1})z^n + \ldots\ldots\ldots$+(a_1 - a_0)z + a_0 = g(z) + a_0 , where g(z) = $-a_n z^{n+1} - (\lambda - 1)a_n z^n + (\lambda a_n - a_{n-1})z^n + \ldots\ldots\ldots$+(a_1 - a_0)z It was shown in [2] that for two complex numbers b_0 , b_1 if $|b_0| \ge |b_1|$and$|arg.b_i - \beta| \le \alpha \le \frac{\pi}{2}$, (i= 0,1)for someβ then $|b_0 - b_1| \le$ ($|b_0|$ - $|b_1|$) $\cos\alpha$ + ($|b_0|$ + $|b_1|$) $\sin\alpha$
Hence for$|z| < 1$,$|g(z)| \le |a_n|$+(λ -1) $|a_n|$ + $|\lambda a_n - a_{n-1}|$ + $|a_{n-1} - a_{n-2}|$ + ...+ $|a_1 - a_0| \le \lambda|a_n| + (\lambda|a_n| - |a_0|)\cos\alpha + (\lambda|a_n| + |a_0|)\sin\alpha + 2\sin\alpha\sum_{i=0}^{n-1}|a_i|$
Again we have , g(0) = 0, therefore by Schwarz lemma we obtain $|g(z)| \le M|z|$for$|z| < 1$, where M = $\lambda|a_n| + (\lambda|a_n| - |a_0|)\cos\alpha + (\lambda|a_n| + |a_0|)\sin\alpha + 2\sin\alpha\sum_{i=0}^{n-1}|a_i|$Therefore for $|z| < 1$, we have$|F(z)|$ = |g(z) $+ a_0| \ge |a_0| - |g(z)| \ge |a_0| - M|z| > 0$, if$|a_0| > M|z|$

i.e, if$|z| < \frac{|a_0|}{M}$, where M = $\lambda|a_n| + (\lambda|a_n| - |a_0|)\cos\alpha + (\lambda|a_n| + |a_0|)\sin\alpha + 2\sin\alpha\sum_{i=0}^{n-1}|a_i|$

Also we can show that M ≥ $|a_0|$ as $|z| < 1$ Hence the desired result follows.

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